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by Peter Hördahl [2], Oreste Tristani [3] and David Vestin [4]

In 2007 all ECB publications feature a motif taken from the €20 banknote.

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CONTENTS

Abstract

We show that microfounded DSGE models with nominal rigidities can be successful in replicating features of bond yield data which have previously been considered puzzling in general equilibrium frameworks. Consistent with empirical evidence, we obtain average holding period returns that are positive, increasing in maturity and sizable, as well as long-maturity bond yields that are almost as volatile as short-term interest rates. At the same time, we are able to fit sample moments of consumption and inflation relatively well. To improve our understanding of these results, we derive analytical solutions for yields that are valid up to a second order approximation and generally applicable. We demonstrate that the improved model performance does not arise directly from the presence of nominal rigidities: *ceteris paribus*, the introduction of sticky-prices in a simple model tends to reduce premia. Sticky prices help indirectly because they imply (short-run) monetary non-neutrality, so that the policy rule followed by the central bank affects consumption dynamics and the pricing of yields. A very high degree of "interest rate smoothing" in the policy rule is essential for our results.

JEL classification: E43, E44

Keywords: DSGE models, policy rules, term structure of interest rates, risk premia, second order approximations.

Non-technical summary

This paper analyses the term structure implications of a relatively standard micro-founded macro model featuring nominal rigidities, inflation persistence and habit formation in consumption. The model is solved using second order perturbation methods to allow for a meaningful role for risk. Previous work in this field has concluded that macroeconomic models are hard to reconcile with observed features in financial markets, including the behaviour of the term structure of interest rates. We find, in contrast, that with some modifications of the standard framework, it is possible to match key features of the data. More specifically, we obtain average holding period returns that are positive, increasing in maturity and sizable. Furthermore, long-maturity bond yields implied by the model are almost as volatile as short-term interest rates, in line with the observed behaviour of yields. Importantly, our calibrated model delivers a relatively good fit in terms of features of macroeconomic variables, such as the second moment of inflation and consumption growth. Previous work has been able to reproduce some features of bond yields only at the expense of generating implausible behaviour of macroeconomic variables, such as a counterfactual negative correlation of consumption growth.

To improve our understanding of these results, we derive analytical solutions for yields that are valid up to a second order approximation and generally applicable. We demonstrate that the improved model performance does not arise directly from the presence of nominal rigidities: *ceteris paribus*, the introduction of sticky-prices in a simple model tends to reduce premia. Sticky prices help indirectly because they imply (short-run) monetary non-neutrality, so that the policy rule followed by the central bank affects consumption dynamics and the pricing of yields. A very high degree of "interest rate smoothing" in the policy rule is essential for our results.

We also derive a Fisher-type decomposition of nominal expected excess holding period returns – i.e. the expected excess return from holding a long term nominal bond for one period compared to the return on a 1-period nominal bond – into a real risk premium component and an inflation risk premium component. In our calibration, the average slope of the term structure of nominal interest rates is almost entirely due to real risk premia.

From a technical viewpoint, we derive analytical second order approximate solutions for bond prices consistent with the formulation in Schmitt-Grohé and Uribe (2004). This allows us to solve the macroeconomic model first, and compute yield prices in a second stage. Our expressions also allow one to easily derive holding premia in any DSGE model where bonds do not affect the macroeconomic equilibrium, based on the sole knowledge of the log-linearised solution for inflation and the marginal utility of consumption.

While improving on previous results, our model does not provide a fully satisfactory characterisation of all unconditional and conditional moments of yield data. For example, it does not deliver any time-variation in risk premia, a result which is inconsistent with much empirical work in the financial literature. In order to allow for such time-variation, higher-order approximation methods must be employed.

1 Introduction

At the core of dynamic macroeconomic models, we find equilibrium relationships which describe the allocation of quantities (e.g. consumption and investment) and, when imperfect competition prevails, the setting of prices. These equations are typically valid across time and across states of nature, and financial assets are the tool that are supposed to ensure their validity. Financial assets are therefore an integral part of macroeconomic models. Any well-specified model should be able to match financial data as well as macro data.

It is therefore problematic that microfounded models have had a hard time explaining key features of asset prices. The most famous example of this difficulty is the equity premium puzzle, but various features of bond yield data have also been characterised as puzzling in the literature. A number of papers, including Backus, Gregory and Zin (1989), Donaldson, Johnsen and Mehra (1990), Den Haan (1995) and Chapman (1997), have concluded that general equilibrium models cannot generate term premia of a magnitude comparable to what we observe in actual data – and may not even be capable of producing positive term premia.

For equity, one could argue that fundamentals (the expected future profitability of individual firms) are unobservable and difficult to evaluate. Equity prices may therefore be thought to be subject to fluctuations disconnected from the real economy (information acquisition, fads etc.), and one could hope that the inability of macroeconomic models to match equity prices does not represent a signal of misspecification. This argument is more difficult to construct for bonds. Bond yields should ultimately reflect expectations of future monetary policy decisions which, at least in recent years, are arguably more predictable than equity fundamentals. Thus, the ability of macroeconomic models to explain bond prices represents an important test of their empirical performance.

In this paper, we revisit the relationship between bond prices and macroeconomic fundamentals. Contrary to the papers cited above, that rely on flexible price models, we investigate whether nominal rigidities are helpful to match yields data within general equilibrium models. Our interest in models with nominal rigidities arises from the fact that versions of these models have been shown to describe macro data with some degree of success.

The broad flavour of our results is that relatively complex models, such as those incorporating nominal rigidities, are closer to yield data than one would conclude based on previous analyses. The benchmark version of the model which we employ is able to approximate well some key average features of yield data, while remaining broadly consistent with the sample moments of macroeconomic data. The model includes just two persistent sources of exogenous uncertainty: a technology shock, predominant in size, and an inflation target shock.

The improved model performance, however, does not arise directly from the presence of nominal rigidities. On the contrary, we demonstrate that the market prices of risk, a fundamental determinant of term premia, are always smaller in a plain-vanilla model with nominal rigidities than in a corresponding model with flexible prices. Nominal rigidities help indirectly, because they imply that monetary policy has real effects. The characteristics of the policy rule followed by the central bank become an important determinant of the dynamics of consumption and, in turn, the term structure of risk premia. More specifically, the improved performance of our model is crucially determined by a high degree of "interest rate smoothing" – i.e. the desire to keep interest rate volatility low – in the monetary policy rule.

Our approach allows us to largely resolve three puzzles with respect to features of bond yields within microfounded models, namely the reported inability of such models to generate: (1) positive serial correlation in consumption growth and a positive slope of the term structure simultaneously; (2) roughly constant volatility of bond yields along the term structure; and (3) sizeable term premia. Concerning the first of these puzzles, initially highlighted by Backus, Gregory and Zin (1989), we show that it can be related to the persistence of impulse responses of consumption to shocks in the model. More precisely, consumption must return to its long run equilibrium more slowly than if it followed a simple autoregressive process. We show that various mechanisms can generate this result in our framework, thereby resolving the first puzzle. The second puzzle – i.e. the difficulty for DSGE models to generate sufficiently high volatility in long-term bond yields – disappears if the persistence of exogenous shocks is sufficiently high. Our contribution here is to show that this is possible without causing implausible implications for the moments of consumption and inflation. Finally, concerning the third puzzle, we show that all features of our model can generate term premia of the same order of magnitude as in the data.

To improve our intuition for these results we derive analytical second order approximate solutions for bond prices. Rigorous mathematical foundations for various methods that approximate the solution of nonlinear DSGE models have been presented by Kenneth Judd in a number of contributions (see Judd, 1998, and the references therein). Here, we follow the formulation proposed in Schmitt-Grohé and Uribe (2004). Our analytical solutions are generally applicable to any DSGE model where bonds are priced recursively.

Our paper is related to a growing literature on the implications for nominal bond prices of microfounded models with nominal rigidities. Wu (2006) shows that the approximate bond pricing framework arising from these models is consistent with so-called "affine term structure models" – see Duffie and Kan (1996) and Dai and Singleton (2000). Wu relies on a mixed loglinear-lognormal solution approach, namely an approach whereby the macro-model is solved by (log)linearisation methods, while bond prices are found using second order approximations. Bekaert, Cho and Moreno (2006) and Gallmeyer, Hollifield and Zin (2005) aslo rely on the loglinear-lognormal solution approach. In contrast, we apply second-order approximations consistently throughout our model. Ravenna and Seppälä (2005) study a model similar to ours using third-order approximations, but they rely entirely on numerical methods. Buraschi and Jiltsov (2005, 2007) examine the term structure of interest rates in general equilibrium, but focus on flexible-price models. Wachter (2006) and Piazzesi and Schneider (2006) study the nominal term structure of interest rates, but they model inflation and consumption as exogenous processes in a partial equilibrium setting, in the tradition of e.g. Boudoukh (1993) and Cox, Ingersoll and Ross (1985).

The paper is organised as follows. Section 2 summarises a few widely-known stylised facts on the term structure of interest rates, which we use as a benchmark for the evaluation of the theoretical model. The model is described in Section 3, while Section 4 presents the solution method and the analytical second-order approximation to bond prices. Section 5 shows the quantitative performance of the model and presents our main results. Sections 6, 7 and 8 go on to provide more intuition on the model's performance with respect to three bond price puzzles reported in the literature. Section 9 concludes.

2 Stylised facts of bond yields: a quick review

This section summarises some stylised facts on the term structure of interest rates. These are well known from previous studies, but we briefly summarise them again with reference to an updated dataset of zero-coupon rates provided by the Federal Reserve Board.

Table 1: Summary yields statistics: 1961Q2 - 2007Q2.

Maturity	3m	6m	1y	3y	5y	10y
Mean	1.47	1.49	1.53	1.63	1.68	1.76
Std.Dev.	0.72	0.71	0.70	0.66	0.64	0.60
Autocorr.	0.92	0.93	0.94	0.95	0.96	0.97
ex-post *xhpr*	–	0.04	0.11	0.23	0.28	0.33

Quarterly US data, in percent. Source: Federal Reserve Board. Ex-post excess holding period returns are calculated using continuously-compounded yields.

Looking at US nominal bond yields since the beginning of the 1960s, a number of key features become apparent (see Table 1). First, on average, the term structure has been upward-sloping, with the (annualised) mean of the 10-year yield exceeding the mean of the one-quarter interest rate by 116 basis points over the 1961Q2-2007Q2 period (note that the table reports quarterly figures). Second, the term structure of yield volatilities has tended to be only slightly downward-sloping; the standard deviation of short-term interest rates (up to about one year) was around 2.9% while the corresponding value for the 10-year yield was around 2.4%. Third, yields have been highly persistent. The first-order sample autocorrelations of quarterly yields were above 0.9 across all maturities. Fourth, realised excess holding period returns (henceforth, *xhpr*'s)[1] on bonds tended to be increasing in the maturity of the bonds. For example, while the average return earned from investing in a 6-month bond for one quarter, in excess of the 3-month interest rate, was around 16 basis points (in annualised terms), this excess return increased up to 112 basis points for 5-year bonds (also annualised). This would suggest that *expected* excess returns should also be increasing with maturity.

It could be argued that these unconditional moments should be calculated over a shorter sample period, because of the impact on the yield curve of possible structural breaks. Many authors, for example, have argued that a break in the Fed's monetary policy rule took place most probably in 1979 (see e.g. Clarida, Galí and Gertler, 2000, and

[1] Specifically, we take *xhpr* to denote the *expected* excess holding period return when discussing properties of our model, whereas when discussing data we refer to the average realised excess holding period return.

Orphanides, 2004). The so-called "monetarist experiment" of the initial years of Paul Volcker's tenure as Federal Reserve Chairman is also assumed to represent a specific regime by some authors. An analysis of the data during these subperiods reveals that the broad qualitative features of the term structure data are robust, even if some quantitative variations are evident, especially concerning ex-post excess returns.[2] We focus on full-sample statistics because of the need for many observations to identify unconditional means of highly persistent variables.

3 The model

The model we employ can be seen as a simplified version of that proposed in Christiano, Eichenbaum and Evans (2005). It also borrows heavily from Woodford (2003). The central feature is the assumption of nominal rigidities, which generates real effects of nominal shocks. The specification of the monetary policy rule therefore becomes important in affecting economic dynamics. The real side of the model features a continuum of households consuming a basked of goods, and providing labour services to firms. The firms are monopolistic competitors and only use labour for production. The solution of the model pins down the behavior of the stochastic discount factor, which, in turn, is used to price nominal and real bonds, hence allowing us to recover the term structure.

Models such as those proposed in Christiano, Eichenbaum and Evans (2005) also include a large number of exogenous shocks. Here we focus on fewer shocks, which are selected to provide a general illustration of the empirical performance of the model.

3.1 Consumers

Consumers maximise the discounted sum of the period utility

$$U\left(C_t, H_t, L_t\right) = \frac{\left(C_t - H_t\right)^{1-\gamma}}{1-\gamma} - \int_0^1 \chi L_t\left(i\right)^{\phi} di \tag{1}$$

[2] Specifically, while ex-post excess holding period returns were close to zero on average, or even negative for longer maturities, during the 1960-1978 period, they were large and positive on average in the second part of the sample. The large time variability of $xhpr$'s is well known in finance-type studies – see e.g. Dai and Singleton (2002) – and a puzzle by itself for microfounded models. We do not explore this aspect of the data here, since second order approximations yield constant excess returns by construction. Other aspects of the data, such as yield standard deviations, were much more stable in different sub-periods.

where C is a consumption index satisfying

$$C = \left(\int_0^1 C(i)^{\frac{\theta-1}{\theta}} di \right)^{\frac{\theta}{\theta-1}}, \tag{2}$$

$H_t = hC_{t-1}$ is the habit stock and, by assumption, workers provide $L(i)$ hours of labor to firm i.[3] For consistency with Smets and Wouters (2003) and Christiano, Eichenbaum and Evans (2005), habit formation is modelled in difference form. Moreover, habit is internal, so that households care about their own lagged consumption. We introduce habit persistence because it has been shown to improve the empirical performance of models, both in macroeconomics (e.g. Fuhrer, 2000) and in finance (e.g. Constantinides, 1990; Campbell and Cochrane, 1999; Dai, 2003).

The households' budget constraint is given by

$$P_t C_t + S_t \leqslant \int_0^1 w_t(i) L_t(i)\, di + \int_0^1 \Xi_t(i)\, di + W_t \tag{3}$$

with the price level P_t defined as the minimal cost of buying one unit of C_t

$$P_t = \left(\int_0^1 p(i)^{1-\theta} di \right)^{\frac{1}{1-\theta}}. \tag{4}$$

In the budget constraint, S_t denotes end-of-period holdings of a complete portfolio of state contingent assets (this is not to be confused with the lower-case s used as a running index in the infinite sums below). W_t denotes the beginning-of-period value of the assets, $w_t(i)$ is the nominal wage rate, $L_t(i)$ is the supply of labor to firm i and $\Xi_t(i)$ are the profits received from investment in firm i.

Finally, complete markets imply the existence of a unique pricing kernel (or stochastic discount factor, or state price deflator), denoted $Q_{t,t+1}$. This implies that $S_t = E_t(Q_{t,t+1} W_{t+1})$.

The first order conditions w.r.t. labour supply and intertemporal aggregate consumption allocation are

$$\begin{aligned} \frac{w_t(i)}{P_t} &= \frac{\phi \chi L(i)^{\phi-1}}{\Lambda_t} \\ Q_{t,t+1} &= \beta \frac{P_t}{P_{t+1}} \frac{\Lambda_{t+1}}{\Lambda_t}, \end{aligned} \tag{5}$$

[3] Note that the Frisch elasticity of labour is $1/(\phi - 1)$. The elasticity of intratemporal substitution between goods, θ, should be strictly greater than 1.

where we define the marginal utility of consumption as

$$\Lambda_t \equiv (C_t - hC_{t-1})^{-\gamma} - \beta h \mathrm{E}_t \left[(C_{t+1} - hC_t)^{-\gamma} \right]. \tag{6}$$

Note that the gross interest rate, I_t, equals the conditional expectation of the stochastic discount factor, i.e.

$$I_t = \beta^{-1} \left\{ \mathrm{E}_t \left[\frac{P_t}{P_{t+1}} \frac{\Lambda_{t+1}}{\Lambda_t} \right] \right\}^{-1}. \tag{7}$$

3.2 Firms

Turning to the firms, the production function is given by

$$\begin{aligned} Y_t(i) &= A_t L(i)^{\alpha} \\ A_t &= A_{t-1}^{\rho_A} e^{\varepsilon_t^a} \end{aligned} \tag{8}$$

where A_t is a technology shock and ε_t^a is a normally distributed shock with constant variance σ_a^2.

We assume Calvo (1983) contracts, so that firms face a constant probability ζ of being unable to change their price at each time t. Firms will take this constraint into account when trying to maximise expected profits, namely

$$\max_{P_t^i} \mathrm{E}_t \sum_{s=t}^{\infty} \zeta^{s-t} Q_{t,s} \left(P_s^i Y_s^i - TC_s \right), \tag{9}$$

where TC denotes total costs and, as in Smets and Wouters (2003), firms not changing prices optimally are assumed to modify them using a rule of thumb that indexes them partly to lagged inflation and partly to steady-state inflation $\overline{\Pi}$, namely $P_t^i \left(\overline{\Pi}^{s-t} \right)^{1-\iota} \left(\frac{P_{s-1}}{P_{t-1}} \right)^{\iota}$, where $0 \leq \iota \leq 1$. We adopt the assumption of inflation indexation because it is necessary to generate inflation persistence, a feature of aggregate inflation data which is not present in the standard Calvo-pricing model.

The first-order condition for this problem have firms setting prices as a markup on current and expected future marginal costs. The Dixit-Stiglitz preferences implies a demand function for the firm of the form[4] $Y_t^i = \left(P_t^i / P_t \right)^{-\theta} Y_t$. Output is demand-determined, so that firms sell whichever quantity consumers demand at the set price. Since there is only one factor of production, the labour demand can be found by inverting the production

[4] Note there that we define aggregate output in the same fashion as aggregate consumption, $Y = \left(\int_0^1 y(i)^{\frac{\theta-1}{\theta}} di \right)^{\frac{\theta}{\theta-1}}$.

function. The marginal cost will be given by differentiating this expression with respect to output and multiplying it by the wage rate (which is taken as given by the producer).

Under the assumption that firms are perfectly symmetric in all respects other than the ability to change prices, firms that do get to change their price will set it at the same optimal level P_t^*. Furthermore, the average level of prices in the group that does not change prices is partly indexed to the average price level from the last period (since, by a law of large number type of argument, those firms are drawn from the pool of all firms). We can characterise firms' decisions as implying[5]

$$\begin{aligned}\left(\frac{P_t^*}{P_t}\right)^{1-\theta\left(1-\frac{\phi}{\alpha}\right)} &= \frac{\phi\chi\theta}{\alpha\left(\theta-1\right)}\frac{K_{2,t}}{K_{1,t}} \\ K_{2,t} &= \frac{A_t^{-\frac{\phi}{\alpha}}}{\Lambda_t}Y_t^{\frac{\phi}{\alpha}} + \mathrm{E}_t\zeta\overline{\Pi}^{-\theta\frac{\phi}{\alpha}(1-\iota)}Q_{t,t+1}K_{2,t+1}\Pi_t^{-\theta\frac{\phi}{\alpha}\iota}\Pi_{t+1}^{1+\theta\frac{\phi}{\alpha}} \\ K_{1,t} &= Y_t + \mathrm{E}_t\zeta\overline{\Pi}^{(1-\theta)(1-\iota)}Q_{t,t+1}K_{1,t+1}\Pi_t^{(1-\theta)\iota}\Pi_{t+1}^{\theta}\end{aligned} \tag{10}$$

where Π_t is the inflation rate defined as $\Pi_t \equiv \frac{P_t}{P_{t-1}}$ and

$$\frac{P_t^*}{P_t} = \left(\frac{1-\zeta\left(\overline{\Pi}^{1-\iota}\frac{\Pi_{t-1}^{\iota}}{\Pi_t}\right)^{1-\theta}}{(1-\zeta)}\right)^{\frac{1}{1-\theta}} \tag{11}$$

expresses the optimal price at time t as a function only of aggregate variables.

3.3 Monetary policy and the natural rate of output

We close the model with a simple Taylor-type policy rule

$$i_t = -\left(1-\rho_I\right)\log\beta + \delta\left(\pi_t - \pi_t^*\right) + \kappa\left(y_t - y_{t-1}\right) + \rho_I i_{t-1} \tag{12}$$

$$\pi_t^* = \rho_\pi \pi_{t-1}^* + \varepsilon_t^{\pi^*} \tag{13}$$

where i_t is the logarithm of the gross nominal interest rate (which is riskless in nominal terms), π_t^* is the inflation target and $\varepsilon_t^{\pi^*}$ is a white noise shock with variance $\sigma_{\pi^*}^2$.

Variants of rule (12) have been found to be empirically plausible by a number of authors, including Clarida, Galí and Gertler (2000), Smets and Wouters (2005) and Hördahl, Tristani and Vestin (2006).

[5] Similar recursive expressions are derived, for example, in Ascari (2004) and Benigno and Woodford (2005), for the case without indexation.

We assume that the long run inflation target is zero in order to better isolate the properties of the model in terms of risk premia. A non-zero long-run inflation target would obviously have an impact, in particular, on the spread between nominal and real yields. However, it would not change the risk properties of the model.

3.4 Market clearing and equilibrium

Market clearing equates supply with demand both in product and labour markets. Given the symmetric definitions of aggregate consumption and output together with the absence of government spending and investment, coupled with the fact that all assets must be in zero net supply as there is no idiosyncratic difference between agents in the economy, we obtain

$$Y_t = C_t. \tag{14}$$

Once aggregate output is determined, individual production levels follows from the demand functions, and the amount of labour needed to produce such levels follows from inverting the production function. Finally, the wage rate needed to attract that level of hours follows from (5). Since all we need in order to price bonds is the solution for the aggregate variables, (7), (10), (11), (12) and (14) together with the exogenous dynamics for the shocks (8) and (13) defines a rational expectations system whose solution we now proceed to characterise.

4 Solving the model

We solve the model relying on the second order approximation of the nonlinear relationships which link (the logarithm of) all endogenous variables to (the logarithm of) the predetermined variables. The point around which the approximation is computed is the non-stochastic steady state of the model (see the appendix for further details).

In the rest of the paper, lower case letters will denote the (natural) logarithm of corresponding upper case letters. The model dynamics will then be described by two equations: a quadratic law of motion for the predetermined variables of the model and a quadratic relationship linking each non-predetermined variable to the predetermined variables.

4.1 Macro-model solution

The solution of the macromodel is obtained numerically using the approach (and the routines) described in Schmitt-Grohé and Uribe (2004). For the vector $\widehat{x}_t$ of predetermined variables, we obtain the (partly endogenous) law of motion

$$\widehat{x}_{t+1} = c_1\widehat{x}_t + \frac{1}{2}\begin{bmatrix} \cdots \\ \widehat{x}_t' c_2 [i] \widehat{x}_t \\ \cdots \end{bmatrix} + \frac{1}{2}c_0\sigma^2 + \eta\sigma\varepsilon_{t+1} \tag{15}$$

where c_1, c_0 and η are $n_x \times n_x$, $n_x \times 1$, and $n_x \times n_\varepsilon$ matrices (n_ε the size of the shock vector ε_{t+1}), respectively, and $c_2 [i]$ are symmetric $n_x \times n_x$ matrices (the index i underlines that there are n_x different matrices, one for each predetermined variable). For each non-predetermined variable, we obtain a similar expression. More specifically, for the marginal utility of consumption and inflation we obtain

$$\widehat{\lambda}_t = \lambda_x' \widehat{x}_t + \frac{1}{2}\widehat{x}_t' \lambda_{xx} \widehat{x}_t + \frac{1}{2}\lambda_{\sigma\sigma}\sigma^2 \tag{16}$$

$$\widehat{\pi}_t = \pi_x' \widehat{x}_t + \frac{1}{2}\widehat{x}_t' \pi_{xx} \widehat{x}_t + \frac{1}{2}\pi_{\sigma\sigma}\sigma^2 \tag{17}$$

where λ_x, π_x, $\lambda_{\sigma\sigma}$ and $\pi_{\sigma\sigma}$ are vectors, λ_{xx} and π_{xx} are symmetric matrices, σ is the perturbation parameter and $\widehat{x}_t$ is the vector of n_x predetermined variables.

Note that the nominal stochastic discount factor can be written in log-deviation from the deterministic steady state as $\widehat{q}_{t,t+1} = \Delta\widehat{\lambda}_{t+1} - \widehat{\pi}_{t+1}$, where Δ is the first difference operator such that $\Delta\widehat{\lambda}_{t+1} = \widehat{\lambda}_{t+1} - \widehat{\lambda}_t$. Hence, the solution equations above complete the description of the dynamics of the stochastic discount factor.

4.2 Solving for bond prices

Given the stochastic discount factor, bond prices can be derived recursively. The price of an n-period nominal bond, $B_{t,n}$, which at maturity satisfies $B_{t+n,0} = 1$, can be written as $B_{t,n} = \mathrm{E}_t [Q_{t,t+1}B_{t+1,n-1}]$. In the appendix, we show that a second order approximate solution for bond prices, in log-deviation from their deterministic steady state, can be written in the form

$$\widehat{b}_{t,n} = B_{n,x}' \widehat{x}_t + \frac{1}{2}\widehat{x}_t' B_{n,xx} \widehat{x}_t + \frac{1}{2}\sigma^2 B_{n,\sigma\sigma} \tag{18}$$

where the vectors $B_{n,x}$ and $B_{n,\sigma\sigma}$ and the symmetric matrices $B_{n,xx}$ are defined recursively as

$$\begin{aligned} B'_{1,x} &= \left(\lambda'_x - \pi'_x\right) c_1 - \lambda'_x \\ B_{1,\sigma\sigma} &= \left(\lambda'_x - \pi'_x\right) c_0 - \pi_{\sigma\sigma} + \operatorname{tr}\left(\eta' \left(\lambda_{xx} - \pi_{xx}\right) \eta\right) + \left(\lambda'_x - \pi'_x\right) \eta\eta' \left(\lambda'_x - \pi'_x\right)' \\ B_{1,xx} &= c'_1 \left(\lambda_{xx} - \pi_{xx}\right) c_1 - \lambda_{xx} + \sum_{i=1}^{n_x} \left(\lambda'_x [i] - \pi'_x [i]\right) c_2 [i] \end{aligned} \quad (19)$$

and for $n > 1$

$$\begin{aligned} B'_{n,x} &= B'_{1,x} + B'_{n-1,x} c_1 \\ B_{n,xx} &= B_{1,xx} + c'_1 B_{n-1,xx} c_1 + \sum_{i=1}^{n_x} B'_{n-1,x} [i] c_2 [i] \\ B_{n,\sigma\sigma} &= B_{1,\sigma\sigma} + B_{n-1,\sigma\sigma} + B'_{n-1,x} c_0 + \operatorname{tr}\left(\eta' B_{n-1,xx} \eta\right) \\ &\quad + B'_{n-1,x} \eta\eta' B_{n-1,x} + 2 B'_{n-1,x} \eta\eta' \left(\lambda_x - \pi_x\right) \end{aligned} \quad (20)$$

In these expressions, $\lambda_x [i]$, $\pi_x [i]$ and $B_{n-1,x} [i]$ are the $i-$th (scalar) elements of the λ_x, π_x and $B_{n-1,x}$ vectors, and $c_2 [i]$ is the symmetric c_2 matrix which characterises the solution for the $i-$th state variable. In order to derive these expressions we rely on the suggestion in Sutherland (2002) that first order approximate solutions are sufficient to derive second order approximate solutions to second moments.

The above solutions for bond prices are useful because they allow us to develop a better intuition for the determinants of risk premia. We can derive yield moments and the market prices of risk in terms of their macroeconomic driving forces. We exploit these expressions in Section 8 to obtain an intuition for the features of the model that generate sizeable premia. From a practical perspective, our analytical solutions also produce savings in terms of computational time, compared to the alternative of solving for bond prices recursively, maturity by maturity, jointly with the rest of the model.

In addition, our solution equations (19)-(20) allow us to pin down and quantify the bias involved in the mixed loglinear-lognormal solution approach, sometimes used in the literature, whereby the macromodel is solved by loglinearisation methods, while bond prices are solved using second order approximations. For 1-period bonds, for example, this approach amounts to ignoring the second order terms λ_{xx}, π_{xx}, and $\pi_{\sigma\sigma}$ in system (19). As a result, the second order coefficients $B_{n,xx}$ and $B_{n,\sigma\sigma}$ would not be second-order accurate. This would have implications on both the dynamic of yields and on their unconditional means. In our benchmark calibration, the difference in unconditional means turns out to be small. In the loglinear-lognormal approach, average yields are biased negatively (they are too low) by 8 basis points at the 1-quarter maturity, by 7 basis points at the 1-year maturity and by 1 basis point at the 10-year maturity. Therefore, the loglinear-lognormal approach produces a slight underestimation of the unconditional slope of the yield curve.

Finally, it can be noted that the second order approximation of our DSGE model is related to the one of so-called macro-finance models (see e.g. Hördahl, Tristani and Vestin, 2006; Rudebusch and Wu, 2007). The macro-finance approach typically amounts to disregarding any nonlinear terms in the law of motion of the predetermined variables (i.e. setting the matrices $c_2[i]$ and the vector c_0 to zero), as well as quadratic terms in the state vector in the bond equations (i.e. setting the matrix $B_{n,xx}$ to zero). Constant terms such as $B_{n,\sigma\sigma}$ are allowed, but not derived consistently with the utility function of a representative agent. Finally, the nominal stochastic discount factor $\widehat{q}_{t,t+1}$ is modelled exogenously in an empirically flexible manner, which is also not derived from the utility of a representative agent.

5 Matching unconditional moments of the yield curve and macroeconomic variables

In this section, we present results based on a calibrated version of our model and argue that it fits unconditional features of both the term-structure and macroeconomic variables reasonably well. The following three sections will examine special cases of the model in order to provide a better intuition for what is driving the results.

Concerning the term structure, we focus on three features considered puzzling within microfounded models: (1) the relationship between the sign of term premia and the serial correlation of consumption growth – a puzzle highlighted by Backus, Gregory and Zin (1989); (2) the "large" volatility of long term yields; and (3) the sizable term premia. We return to these puzzles after the presentation of our benchmark calibration results in this section.

Table 1 showed that both the term structure and average excess holding period returns – i.e. the difference between the average return from holding a bond of maturity n for only 1 quarter and the 3-month rate in the same quarter – are upward-sloping. In the rest of the paper, we focus our analysis of risk premia on the latter measure, namely expected excess holding period returns ($xhpr$'s). The advantage of holding period returns over yields is that the former are not affected by first moments. While the slope in yields can be caused purely by (risk-neutral) expectations of future policy changes over the horizon of the yields, holding period returns have all the same time horizon as the short-term rate. They are the most direct measure of risk premia on bonds, as emphasised by the fact that they are all zero at each point in time and for all maturities in the first order approximation of a model.

In terms of macroeconomic variables, we analyse the unconditional moments of consumption growth, inflation and the nominal interest rate.[6] The parameters values used in the calibration, given in Table 2, are relatively standard. We mainly rely on the results of Smets and Wouters (2005), who estimate them on US data over the 1974:1-2002:2 and 1983:1-2002:2 periods and find them to remain relatively stable. Many parameters are also consistent with those used by Woodford (2003).

Table 2: Benchmark calibration

α	θ	γ	ϕ	h	ι	ζ
0.76	7.88	6	3.45	0.69	0.66	0.63
δ	κ	ρ_I	ρ_A	ρ_π	σ_A	σ_π
1.85	0.0	1.0	0.995	0.999	0.0332	0.0002

Note: Other parameter values are: $\beta = 0.99$; $\chi = 1/\phi$.

More specifically, the weight of labour in the production function is $\alpha = 0.76$, the between-good elasticity θ corresponds to a steady state mark-up of about 15%, and the degree of habit persistence is $h = 0.69$. We set indexation $\iota = 0.66$, while the discount factor is $\beta = 0.99$, as is common in the literature. The Frisch elasticity of labour is around 0.4. We experimented with two values of the risk aversion parameter: $\gamma = 1.6$, which is Smets and Wouters' estimated value, and $\gamma = 6$, which is the valued estimated by Fuhrer (2000). Since habits modify the relationship between the intertemporal rate of substitution of consumption and risk aversion, we take Fuhrer's estimate as an example

[6] The macro data are taken from the FRED database of the Federal Reserve Bank of St. Louis. We use PCECC96 for consumption and PCECTPI (the personal consumption deflator) for prices. These series are available from 1947. For our calibration exercise, we focus on data starting in 1961 for consistency with our yields dataset.

of high but still reasonable value of γ.[7] We only report simulation results based on $\gamma = 6$ in the paper. For the persistence of the shocks, we choose Smets and Wouters' estimated values of $\rho_A = 0.995$ and $\rho_\pi = 0.999$ (Smets and Wouters actually allow for a unit root in the inflation target). We use a value for the probability of firms not changing their price $\zeta = 0.63$, which is lower than Smets and Wouters' estimate, but more in line with standard calibrations.

Finally, we perform a rough grid search across the values of the policy rule parameters – the inflation and output gap reaction coefficients – and the standard deviations of the two exogenous shocks, to maximise the match between observed and calibrated moments. The results are an inflation reaction coefficient $\delta = 1.85$, an output gap reaction coefficient $\kappa = 0$ and an interest rate smoothing coefficient in the policy rule $\rho_I = 1.0$. The rule is therefore a "difference rule" of the sort advocated, for example, in Orphanides and Williams (2002). The standard deviations of the shocks turn out to be $\sigma_A = 0.0332$, a high value compared to standard RBC-type calibrations, and $\sigma_\pi = 0.0002$.

Table 3: Unconditional moments and excess holding period returns

	US 1961:3-2007:2	Benchmark	$h = 0$	$\zeta = 0$	$\rho_I = 0$
Var $\left[\widehat{ytm}_{20}\right]$	**0.40**	*0.40*	0.40	0.47	1.92
Var $\left[\widehat{i}\right]$	**0.51**	*0.48*	0.41	2.67	2.22
Var $[\widehat{\pi}]$	**0.39**	*0.20*	0.20	0.66	1.32
$xhpr_{20}$	**0.28**	*0.18*	0.07	0.47	0.26
Var $[\Delta\widehat{c}]$	**0.44**	*0.51*	2.43	0.56	0.47
Cov $[\Delta\widehat{c}, \widehat{\pi}]$	**−0.15**	*−0.03*	−0.01	−0.23	−0.12
Cov $\left[\Delta\widehat{c}, \widehat{i}\right]$	**−0.10**	*−0.14*	0.02	−1.08	−0.22
Cov $\left[\widehat{i}, \widehat{\pi}\right]$	**0.29**	*0.20*	0.20	0.63	1.63
$Cor[\Delta\widehat{c}, \Delta\widehat{c}_{+1}]$	**0.23**	*0.67*	0.00	0.62	0.68

Note: quarterly figures; unconditional variances and covariances are $\times 10^4$. A value of 0.00 denotes a quantity smaller than 0.005.

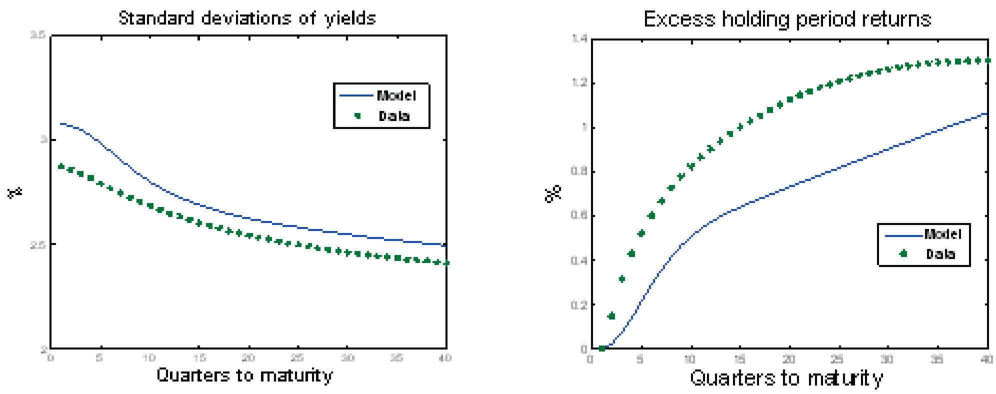
[7] The main unattractive implications of a high degree of risk aversion is that, with standard preferences, the elasticity of intertemporal substitution of consumption, $1/\gamma$, becomes unreasonably low. With habit persistence, however, intertemporal substitution is larger than $1/\gamma$.

The results of this simulation in terms of various data moments are summarised in Table 3. The columns denoted as "benchmark" shows that two exogenous shocks are sufficient to make our model broadly consistent with the sample features of the data analysed here. The technology shock alone exerts a major impact on the dynamics of the variables and drives most of the results. The main role of inflation target shocks is to generate some volatility in inflation. By and large, the fit of data moments is reasonably good, even if not perfect. For example, all covariances and the variances of all nominal variables are matched extremely well; moreover, the 5-year expected excess holding period return is sizable. However, the serial correlation of consumption growth is larger, and the variance of inflation smaller, than in the data.

The other columns of the Table show some special cases where specific features of the model are "switched off". The column labeled "$h = 0$" focuses on the case without habit formation (all other parameters are kept as in the benchmark column). This special case demonstrates that the high serial correlation of consumption growth is directly linked with the habit parameter. It also shows that, without habits, excess holding period returns would be much smaller and, at the same time, the variance of consumption growth would increase almost by a factor of 5, compared to the benchmark calibration. The columns labeled "$\zeta = 0$" analyses the implication of removing price stickiness (again, all other parameters are kept as in the benchmark column). In this case, it would actually be easier to match the slope of the yield curve, but only at the cost of a five-fold increase in the variance of the short-term rate and of a much steeper decline in the variance of yields. The covariance between consumption growth and the short rate would also be implausibly large in absolute value. Finally, the column labeled "$\rho_I = 0$" focuses on a simpler policy rule, which is more similar to the original suggested by Taylor (1993) (once again, all other parameters are kept as in the benchmark column). In this case, there is a four-fold increase in the variances of the short-rate and the 5-year yield, a six-fold increase in the variance of inflation, and a 5-fold increase in the covariance between inflation and the nominal interest rate.

We draw two main conclusions from Table 3. The first is that the assumption of price-stickiness does not, *per se*, improve our model's ability to match the data. However, price-stickiness implies that monetary policy has real effects, so that the precise specification of the monetary policy rule becomes important to determine economic dynamics. The second conclusion is that all features of the model contribute to generate the benchmark results. More specifically, the hypothesis of habit persistence helps to generate high term premia, while the assumption of interest rate smoothing in the policy rule ensures that habits do not translate into excessively high volatility of short rates. Simplified versions of the model preserve good properties along some dimensions, but tend to do very badly in other dimensions.

Figure 1: Yield standard deviations and excess holding period returns (annualised)



Note: Parameter values as in the benchmark column in Table 3.

The standard deviation of yields and the excess holding period returns for bonds of various maturities are shown in Figure 1, which focuses on the benchmark calibration. The figure confirms that the model is broadly in line with the data. The model-implied volatility of all yields is very close to the data and, more specifically, it follows the same declining pattern. Theoretical expected excess holding period returns are positive and follow broadly the shape of their realised counterparts, even if they are generally a bit lower than in the data.

We now proceed to analyse in more detail, and using analytical results, how our model can solve the Backus, Gregory and Zin puzzle, generate a relatively flat variance of yields across maturities, and produce large excess holding premia.

6 Solving the Backus, Gregory and Zin puzzle

If we define $\widehat{ytm}_n$ and $\widehat{ytm}_n^{\text{real}}$ as the yield to maturity on an n-period nominal and real bond, respectively, $\widehat{i}$ as the short term nominal interest rate and $\widehat{r}$ as the real interest rate (all in deviation from the nonstochastic steady state), the unconditional slope of the term structure of interest rates can be written as[8]

$$\begin{aligned}\mathrm{E}\left[\widehat{ytm}_n - \widehat{i}\right] &= \mathrm{E}\left[\widehat{ytm}_n^{\text{real}} - \widehat{r}\right] - \frac{1}{2}\left(\frac{\mathrm{E}\left[\mathrm{Var}_t\left[\sum_{i=1}^n \widehat{\pi}_{t+i}\right]\right]}{n} - \mathrm{E}\left[\mathrm{Var}_t\left[\widehat{\pi}_{t+1}\right]\right]\right) \\ &\quad + \left(\frac{\mathrm{E}\left[\mathrm{Cov}_t\left[\sum_{i=1}^n \widehat{\pi}_{t+i}, \Delta^n \widehat{\lambda}_{t+n}\right]\right]}{n} - \mathrm{E}\left[\mathrm{Cov}_t\left[\widehat{\pi}_{t+1}, \Delta\widehat{\lambda}_{t+1}\right]\right]\right)\end{aligned} \tag{21}$$

where $\mathrm{E}\left[\widehat{ytm}_n^{\text{real}} - \widehat{r}\right] = -\frac{1}{2}\left(\frac{1}{n}\mathrm{E}\left[\mathrm{Var}_t\left[\Delta^n \widehat{\lambda}_{t+n}\right]\right] - \mathrm{E}\left[\mathrm{Var}_t\left[\Delta\widehat{\lambda}_{t+1}\right]\right]\right)$ is the slope of the real term structure. Note that, in the special case with CRRA utility, the real slope becomes

$$\mathrm{E}\left[\widehat{ytm}_n^{\text{real}} - \widehat{r}\right]\Big|_{h=0} = -\frac{1}{2}\gamma^2\left(\frac{1}{n}\mathrm{E}\left[\mathrm{Var}_t\left[\Delta^n \widehat{c}_{t+n}\right]\right] - \mathrm{E}\left[\mathrm{Var}_t\left[\Delta\widehat{c}_{t+1}\right]\right]\right). \tag{22}$$

The last expression is used by den Haan (1995, equation (3)) to illustrate the puzzle first highlighted by Backus, Gregory and Zin (1989). In the data, the first-order autocorrelation of consumption growth is positive. Intuitively, this should imply that the variance of the rate of growth of consumption over long periods of time will be larger than the variance of the 1-steap ahead growth rate. As a result, the difference $\frac{1}{n}\mathrm{E}\left[\mathrm{Var}_t\left[\Delta^n \widehat{c}_{t+n}\right]\right] - \mathrm{E}\left[\mathrm{Var}_t\left[\Delta\widehat{c}_{t+1}\right]\right]$ should be positive, which implies that the model predicts a negative slope for the real yield curve. In the data, however, the average yield curve is upward sloping. The puzzle is therefore that the model appears to be incapable of generating, at the same time, a positive term structure slope and a positive serial correlation of consumption.

The relationship between term premia and the variance of consumption growth arises from the property of simple models that term premia arise mainly as a result of precautionary savings. Compared to a world of certainty, shocks, e.g. technology shocks, imply that the future consumption stream is uncertain. Since utility is concave in output (consumption), expected output at any future period is always smaller than its certainty equivalent. As a result, all households try to save more (the precautionary savings effect) and delay consumption to the future. Since this is infeasible in equilibrium, the return on real bonds must fall to discourage savings.

The positive slope of the term structure is due to the fact that short term rates fall more, when compared to the certainty equivalent world, than long term yields. This result

[8] This expression corresponds to the standard one obtained under the assumption of lognormal shocks in partial equilibrium models.

requires the effect of shocks on consumption to die out over time, so that consumption far out in the future is less risky, and the related precautionary savings motive less strong, than consumption in the near future.

In the next sub-sections we show how the link between the sign of the yield curve slope and that of the serial correlation of consumption growth can be broken in most models beyond the plain-vanilla RBC framework. This conclusion can be reached already in a model ignoring the monetary side of the economy. We therefore proceed in two steps. First, we illustrate our point within a real framework. Second, we emphasise how it generalises to a monetary environment.

6.1 The real term structure

Equation (22) suggests that a model with stationary real shocks, such as the one used here, will succeed in producing a positive yield curve slope, because it will imply that the variance of consumption growth over the distant future is small. The difficulty for a stationary model is to generate a positive serial correlation of consumption growth.

An intuitively straightforward manner to make consumption growth positively serially correlated in a simple RBC model is to assume permanent shocks to the rate of growth of consumption. However, den Haan (1995) shows that this assumption often makes the steady state yield curve negatively sloped. Intuitively, permanent shocks make consumption far ahead in the future very uncertain, notably much more uncertain than 1-step ahead consumption. As a result, compared to a certainty equivalent world, a precautionary motive will tend to depress yields on long-term bonds more than on short-term bonds.

Throughout this paper, we therefore maintain the assumption of stationary shocks and focus on the condition which generates a positive correlation of consumption growth. The appendix shows that, for any model with a vector-ARMA reduced form, such condition is

$$\mathrm{Cor}\left[\Delta\widehat{c}, \Delta\widehat{c}_{+1}\right] = -\frac{1}{2}\left(1 - \frac{\rho_1 - \rho_2}{1 - \rho_1}\right) > 0 \tag{23}$$

where ρ_1 and ρ_2 are the first and second order autocorrelation coefficients of the *level* of consumption.

It is well-known that, in a simple model where consumption follows an AR(1) process, autocorrelations decay geometrically (i.e. $\rho_2 = \rho_1^2$) and the above condition cannot be satisfied for any value of the autocorrelation parameter of the AR(1) process. The inequality in (23) can however be satisfied if consumption follows an ARMA process. Loglinearised DSGE models with endogenous, or latent, predetermined variables also have a vector-ARMA reduced form solution. These models, therefore, turn out to be well-placed to solve the Backus, Gregory and Zin puzzle.

Condition (23) implies that the impulse response of consumption to an exogenous shock must be more persistent than it would be if consumption followed an autoregressive process. The reason why this property leads to a solution of the puzzle is clearest when the impulse response is hump-shaped – a particularly strong type of persistence. A hump-shaped impulse response implies that consumption growth will be positive for a few periods, before turning negative: as a result, consumption growth will also be positively correlated. At the same time, a hump-shaped impulse response implies that shocks eventually die out over time, so that the precautionary savings motive plays a minor role on long term bonds. The average yield curve slope will therefore tend to be positive.

In the next two subsections, we first demostrate analytically how the Backus, Gregory and Zin puzzle disappears as soon as we introduce habit persistence in a very simple RBC-type model. Next, we illustrate numerically how other sources of endogenous persistence are individually capable of solving the puzzle.

6.2 Special case: flexible prices and habit formation

As a benchmark for comparison, we start from the simple case of a model without habit persistence nor price rigidities and only technological shocks.

In this case, the yield to maturity of a bond with maturity n at time t can be characterised as

$$\widehat{ytm}_{n,t} = a_n + b_n \ln A_t \tag{24}$$

where

$$\begin{aligned} a_n &\equiv -\ln\beta - \frac{1}{2n}\frac{1-\rho_A^{2n}}{1-\rho_A^2}\left(\gamma\frac{\frac{\phi}{\alpha}}{\frac{\phi}{\alpha}+\gamma-1}\right)^2\sigma_\varepsilon^2 \\ b_n &\equiv -\frac{1}{n}\left(1-\rho_A^n\right)\gamma\frac{\frac{\phi}{\alpha}}{\frac{\phi}{\alpha}+\gamma-1}. \end{aligned}$$

Note that $\frac{\partial a_n}{\partial n} > 0$, hence term premia are increasing in maturity. The term $(\phi/\alpha)\,/\,(\gamma-1+\phi/\alpha)$ represents the elasticity of consumption (output) to the technology shock, so that we could define the standard deviation of consumption as $\sigma_c \equiv (\phi/\alpha)/(\gamma - 1 + \phi/\alpha)\,\sigma_\varepsilon$. Using this definition, the term structure slope can be characterised in terms of the spread between an n-period and a 1-period bond

$$\widehat{ytm}_{n,t} - \widehat{r}_t = \frac{1}{2}\gamma^2 \left(1 - \frac{1}{n}\frac{1-\rho_A^{2n}}{1-\rho_A^2}\right)\,\sigma_c^2 \tag{25}$$

or in terms of expected excess holding period returns

$$\widehat{hpr}_{n,t} - \widehat{r}_t = \left(1 - \rho_A^{n-1}\right)\gamma^2\sigma_c^2 \tag{26}$$

Equation (25) corresponds to the simplest case considered by den Haan (1995, Section 2.4). Using den Haan's calibration parameters ($\rho_A = 0.95$, $\gamma = 6$ and $\sigma_c = 0.007$), the excess yield on a 5-year bond is slightly positive. We therefore know from condition (23) that the model generates a negative serial correlation of consumption growth.

The Backus, Gregory and Zin puzzle disappears as soon as we introduce habit formation in the simplified model. In this case, the marginal rate of intertemporal substitution is not simply proportional to the rate of consumption growth. The former can then be negative, thus producing a positive term structure slope, even if the latter is positive.

In the model with habits, the explicit solution for equilibrium consumption (to a first-order approximation) is

$$c_t = \tau_2 y_{t-1} + \tau_3 a_t \tag{27}$$

where

$$\begin{aligned}
\tau_2 &\equiv \frac{\omega - \sqrt{\omega^2 - 4\beta\gamma^2 h^2}}{2\beta\gamma h} < 1 \\
\tau_3 &\equiv \frac{(1-\beta h)(1-h)}{\gamma(1-\beta h \rho_A)} \\
\omega &\equiv \gamma\left(1+\beta h^2\right) + (1-\beta h)(1-h)\left(\frac{\phi}{\alpha} - 1\right)
\end{aligned}$$

and $\tau_3 > 0$ and $0 < \tau_2 \leq h$ are functions of the structural parameters.

Once again, with stationary shocks expected excess holding period returns are always positive. We can also show that condition (23) for a positive serial autocorrelation of consumption growth boils down to

$$\tau_2 > \frac{1-\rho_A}{1+\rho_A} \tag{28}$$

To see how this can generate a hump-shaped impulse response function, suppose we are in the steady state ($c_{t-1} = 0$) and that a shock $a_t > 0$ occurs. Then $c_t = \tau_3 a_t$. In the absence of new shocks, $c_{t+1} = \tau_3 (\tau_2 + \rho_A) a_t$, which is larger than c_t if $\tau_2 > 1 - \rho_A$. For $\rho_A > 0$, the condition $\tau_2 > 1 - \rho_A$ implies that $\tau_2 > (1 - \rho_A) / (1 + \rho_A)$ is always satisfied. As a result, a hump-shaped impulse response to shocks ensures that the condition for positive serial correlation of consumption is satisfied.

When $\phi = \alpha = 1$, the solution for the natural rate of output can be simplified further as

$$c_t = hc_{t-1} + \frac{1}{1 - \beta h \rho_A} \frac{(1 - \beta h)(1 - h)}{\gamma} a_t \tag{29}$$

where the condition for a hump-shaped impulse response becomes simply $h > 1 - \rho_A$. In this case, expected excess holding period returns are also described by a particularly simple expression

$$\widehat{hpr}_{t,n} - \widehat{i}_t = \gamma^2 \left(\frac{1 - \rho_A \beta h^2}{1 - \rho_A \beta h} \right)^2 (1 - \rho_A^n) \sigma_A^2 \tag{30}$$

As in the flexible-price model without habits, integrated technology shocks annul *xhpr*'s at all maturities. For stationary technology shocks, however, expected excess returns can be increasing in their persistence. This feature is suggestive of how the more general model used in Section 5 can match average excess holding period returns in the data without necessarily generating a low volatility of yields at the same time.

6.3 The nominal term structure

In a monetary economy, inflation risk premia could generate a positive yield curve slope, even if the real yield curve slope were flat. Equation (21) shows that two components matter in this respect. The first one on the first line of the equation is a convexity term, while the last component is the true inflation risk premium. In general, the sign of these terms will be determined by factors analogous to those discussed with regards to the slope of the real term structure.

More specifically, the convexity term involves the difference between variances of inflation over long periods and 1-step ahead. Since inflation is even more positively serially correlated than consumption growth in the data, this term will tend to be positive – thus weighing negatively on the nominal term structure – in simple models where inflation follows an AR(1) process. As in the case of consumption, a hump-shaped impulse response of inflation to shocks will tend to be associated with a positive contribution of the convexity term to the nominal slope.

As to the inflation risk premium proper, the serial correlation effects discussed so far interact with the sign of the covariance between the rate of change in the marginal utility of consumption and the inflation rate. In the CRRA case, the risk premium boils down to $-\gamma\left(\frac{1}{n}\mathrm{E}\left[\mathrm{Cov}_t\left[\sum_{i=1}^{n}\widehat{\pi}_{t+i},\Delta^n\widehat{c}_{t+n}\right]\right]-\mathrm{E}\left[\mathrm{Cov}_t\left[\widehat{\pi}_{t+1},\Delta\widehat{c}_{t+1}\right]\right]\right)$. Given the persistence of both consumption and inflation, one could expect their covariance over long periods of time to be larger than the 1-steap ahead covariance. In this case, the inflation risk premium will tend to be positive when consumption growth and inflation are negatively correlated. In more general models, however, the 1-step ahead term may prevail and a positive risk premium would require a positive covariance between consumption growth and inflation.

Table 4 presents the results of numerical simulations based on various simplified versions of our baseline model to illustrate how sources of endogenous persistence other than habit formation can lead to a solution of the Backus, Gregory and Zin puzzle. We focus on cases where technology shocks are the only source of exogenous disturbance to the system and analyse the consequences of introducing various sources of endogenous persistence in the model.

Table 4: Simulation results: the serial correlation of consumption growth

	$Cor[\widehat{c},\widehat{c}_{+1}]$	$Cor[\Delta\widehat{c},\Delta\widehat{c}_{+1}]$
data: US 1961:3-2007:2	*1.00*	*0.23*
	Flex prices: $\zeta=0$	
$h=\iota=\rho_I=0$	0.99	-0.002
$h=0.68;\iota=\rho_I=0$	1.00	0.622
	Sticky prices: $\zeta=0.63$	
$h=\iota=\rho_I=0$	0.99	-0.002
$h=0.69;\iota=\rho_I=0$	1.00	0.680
$\iota=0.66;h=\rho_I=0$	0.99	0.002
$\rho_I=1.0;h=\iota=0$	0.99	0.002

*: a value of 1.00 indicates that the correlation coefficient is larger than 0.995.
Note: the othe parameter values are as in the main calibration exercise, except for $\sigma_\pi=0$. The slope of the yield curve is positive in all cases included in the Table.

Once again, the Backus, Gregory and Zin puzzle is replicated in the plain-vanilla model, and solved as soon as we introduce habit formation, irrespective of whether prices are flexible or sticky (see the rows 2 to 5 in the Table).

With sticky prices, the puzzle can be solved even without the assumption of habit persistence. Inflation indexation or interest rate smoothing are also sufficient to generate persistence in the response of consumption to shocks, hence to make the model consistent with the signs of both the yield curve slope and consumption growth.

7 Matching the large variance of yields

As already emphasised, a stylised fact from Table 1 is that the standard deviation of yields at all maturities is not too different from that of the short term rate. The standard deviation of 5-year and 10-year yields is approximately 90% of the standard deviation of the short term rate. Den Haan (1995) argues that matching the standard deviation of long term yields is particularly difficult within a simple flexible-price general equilibrium model. Intuitively, the difficulty is to generate sufficient persistence in the short term rate to ensure that its variability is transmitted almost one-to-one to long term rates.

The problem of inducing sufficiently high volatility in yields can be seen most clearly in the case with one state variable, where (see the appendix)

$$\frac{\text{Std}\left[\widehat{ytm}_n\right]}{\text{Std}\left[\widehat{i}\right]} = \frac{1}{n}\frac{1-\rho^n}{1-\rho} \tag{31}$$

where ρ is the serial correlation of the state variable and n is the maturity of the yield. Clearly, the only way to boost the volatility of long term yields is to induce a near-random walk behavior in the state variable. More precisely, note that $\frac{1-\rho^n}{1-\rho}$ is increasing in ρ and it is therefore maximised for $\rho = 1$. In practice, ρ needs to be very close to 1. The ratio between the standard deviations of the 5-year yield and the 3-month rate would only equal 0.64 for shocks with a serial correlation equal to 0.95. Even with a persistence of the shocks equal to 0.99, the ratio would reach 0.91 for 5-year yields, but only 0.83 at the 10-year horizon.

Hence, our model generates a roughly flat volatility of yields at all maturities through an extremely high persistence of exogenous shocks. In simple models, this assumption is problematic because expected excess holding period returns tend to be decreasing in the persistence of shocks, especially when shocks are near random walks – see e.g. equation (26). This generates a tension between simple models and the data. In the next section, we investigate the determinants of the relatively good performance of the model also in terms of excess holding period returns.

8 Term premia and the market price of risk

A large expected excess return on long-maturity bonds could be generated through two channels: an increase in their non-diversifiable riskiness; or an increase in the premium households require for given amount of risk. Since the variance of yields of different maturities is similar, the first route appears to be less promising. For this reason, in this section we focus on the notion of the market price of risk, developed in the finance literature. We first derive a decomposition of expected excess returns which highlights the role of the price of risk. We then illustrate how price-stickiness and other features of the model affect prices of risk, when compared to a plain-vanilla model with flexible prices.

In order to understand the determinants of expected excess holding period returns, it is first useful to analyse separately the real and nominal versions of such returns. These can be written as (see the appendix)

$$\widehat{hpr}^{\text{real}}_{t,n} - \widehat{r}_t = -\sigma\lambda_x' \left(I - c_1^{n-1}\right)\eta\xi_\lambda \qquad (32)$$

$$\widehat{hpr}_{t,n} - \widehat{i}_t = -\sigma\left(\lambda_x'\left(\mathbf{I} - c_1\right) + \pi_x' c_1\right)\left(I - c_1\right)^{-1}\left(I - c_1^{n-1}\right)\eta\xi \qquad (33)$$

where $\xi_\lambda \equiv -\sigma\eta'\lambda_x$ and $\xi \equiv \sigma\eta'\left(\pi_x - \lambda_x\right)$.

For the nominal expected excess return, the term $\sigma\left(\lambda_x'\left(\mathbf{I} - c_1\right) + \pi_x' c_1\right)\left(I - c_1\right)^{-1}\left(I - c_1^{n-1}\right)\eta$ can also be written as $B_{n-1,x}'\eta\sigma$ and it therefore represents the impact of a standard combination of shocks on the price of a bond with maturity $n-1$. From this viewpoint, this term can be interpreted as the "amount of risk" borne by an investor holding the bond, namely the typical fluctuation which the investor can expect from the bond price. In the financial literature, the vector ξ is referred to as "nominal market price of risk" and it indicates how the amount of risk translates into pricing.

The notion of market price of risk is interesting because it is independent of the specific characteristics of the asset being priced. In our case, the amount of risk for yields is obviously maturity-dependent, while the price of risk is identical for all bonds. The market price of risk represents a general measure of how a certain economic model translates into risk pricing, without the need to consider the specific features of individual assets. We show below that it is also a more comprehensive measure of risk than the curvature of the utility function with respect to consumption (the coefficient γ). Other features of a model can in fact contribute to affect the way risk is priced, as is obviously the case in models with habit persistence, where the coefficient γ does not directly measure risk aversion.

The nominal market price of risk can, in turn, be split into a real price of risk and an inflation price of risk component. The former can be identified from the expected excess real holding period return displayed in equation (32). In this equation, $\lambda_x' \left(I - c_1^{n-1}\right) \eta\sigma$ measures the impact of any exogenous shock on the price of a real bond, and it therefore represents the amount of real risk borne by an investor holding a real bond of maturity $n-1$. The vector ξ_λ measures the relevance of that risk for the utility of the representative household and it can therefore be named "real market price of risk". For nominal bonds, the market price of inflation risk can therefore be defined as the difference between the price of nominal and real risk. From the definition of ξ, the market price of inflation risk is $\xi_\pi \equiv \sigma\eta'\pi_x$.

To a second order approximation, the market prices of risk depend entirely on first-order terms in the reduced form of the model. They can therefore be characterised analytically in a few special cases of our general model.

8.1 The plain vanilla flex-price model and technology shocks

In the case without habits nor inflation indexation and with flexible prices (i.e. $h = \iota = \rho_I = \kappa = \zeta = 0$), the market price of technology risk is

$$\xi_{\lambda|h=\zeta=0} = \gamma \frac{\frac{\phi}{\alpha}}{\frac{\phi}{\alpha} - 1 + \gamma} \sigma_A > 0 \tag{34}$$

In this case, a higher disutility of labour, ϕ, matters because it increases households' aversion to changing their hours worked over time. The more inelastic labour supply, the more any technology shock will be transferred one-to-one to output and consumption. At the same time, this effect requires an elasticity of intertemporal substitution of consumption $(1/\gamma)$ different from 1. A low elasticity of intertemporal substitution is associated with a less elastic response of output to shocks, since it reduces consumers' willingness to change their consumption patterns over time.

The market price of risk increases monotonically in both γ and ϕ. A special case of this model occurs when $\alpha = \phi = 1$, so that both labour supply and the production function are linear in labour. In this case, $\xi_{\lambda|h=\zeta=0,\frac{\phi}{\alpha}=1} = \sigma_A$, so that the market price of risk σ_A irrespective of the coefficient of relative risk aversion.

8.2 The plain vanilla model with sticky prices and technology shocks

In the case without habits and in which prices are sticky and the policy rule only reacts to inflation ($h = \iota = \rho_I = \kappa = 0$), we can look at both prices of real and nominal risk associated with technology shocks. The prices of risk are given by

$$\xi_{\lambda|h=0} = \gamma \frac{\frac{\phi}{\alpha}}{\frac{\phi}{\alpha} - 1 + \gamma\,\Omega}\,\sigma_A \tag{35}$$

$$\xi_{\pi|h=0} = -\,\delta\,\frac{1-\rho_A}{\delta-\rho_A}\,\xi_{\lambda|h=0} \tag{36}$$

where

$$\Omega \equiv \frac{1}{(1-\zeta)\,(1-\beta\,\zeta)}\left[1+\zeta\left(\frac{(1-\rho_A)\,(1-\beta\,\rho_A)\,\theta}{\delta-\rho_A}\left(\frac{\phi}{\alpha}-1\right)-\frac{\delta-1}{\delta-\rho_A}-\beta\,\frac{\delta-\rho_A^2}{\delta-\rho_A}+\beta\,\zeta\right)\right]$$

and we focus on determinate equilibria assuming $\delta > 1$.

It can be demonstrated that $\xi_{\lambda|h=0}$ is always positive for $\rho_A < 1$. More importantly, *the market price of technology risk is always lower in a sticky-price equilibrium than in a flex-price equilibrium.* The intuition for this result is straightforward. The price of real risk depends on the extent to which exogenous shocks generate non-diversifiable volatility in consumption. In a flexible price model, technology shocks will lead to a change in consumption demand and in firms' production levels. In a sticky price model, the same shocks and the same change in the demand for consumption goods will partly be met by changes in firms' prices. In equilibrium, therefore, aggregate consumption will change less and, consequently, the volatility of consumption will decrease. The lower volatility of consumption in sticky price models is associated with a smaller price of technology risk. As a result, the assumption of sticky prices does not help by itself to better match the upward-sloping term structure of interest rates.

The difference between market price of real risk in the flexible and sticky price versions of the model will be crucially affected by the monetary policy rule. More specifically, the price of risk is increasing in the size of the inflation response coefficient of the monetary policy rule. The reason is that a higher δ brings about lower inflation volatility, but higher volatility of consumption. In the limit of an infinite response coefficient, the volatility of real output is maximised and the market price of real risk becomes identical to the flexible-price case. This explains why expected excess holding period returns increase, in Table 3, in the flexible price column "$\zeta = 0$".

At the same time, technology shocks bring about inflation risk in the sticky price model. Since $\delta(1-\rho_A)/(\delta-\rho_A)<1$, and $\partial[\delta(1-\rho_A)/(\delta-\rho_A)]/\partial\delta<0$, $\partial[\delta(1-\rho_A)/(\delta-\rho_A)]/\partial\rho_A<0$, the market price of inflation risk is always smaller than the price of real risk, and it is also decreasing in both ρ_A and δ. It is high for low values of δ, and smaller the more decisive the central bank's response to inflation deviations from target.

The total nominal price of risk $\xi_{h=0}=\xi_{\lambda|h=0}\rho_A(\delta-1)/(\delta-\rho_A)$ in the sticky-price case is also smaller than the price of real risk in the flex-price case.

8.3 The flex-price plain vanilla model with technology shocks and habits

Under flexible prices and with habit persistence (i.e. $\iota=\rho_I=\kappa=\zeta=0$), the real market price of technology risk is

$$\xi_{\lambda|\zeta=0}\equiv\frac{\phi}{\alpha}\frac{1+\beta h^2-\beta h(1+\rho_A)\tau_2}{\beta h(\tau_1-\rho_A)}\sigma_A \tag{37}$$

where τ_2 and ω are defined above and

$$\tau_1\equiv\frac{2\gamma h}{\omega-\sqrt{\omega^2-4\beta\gamma^2h^2}}>1$$

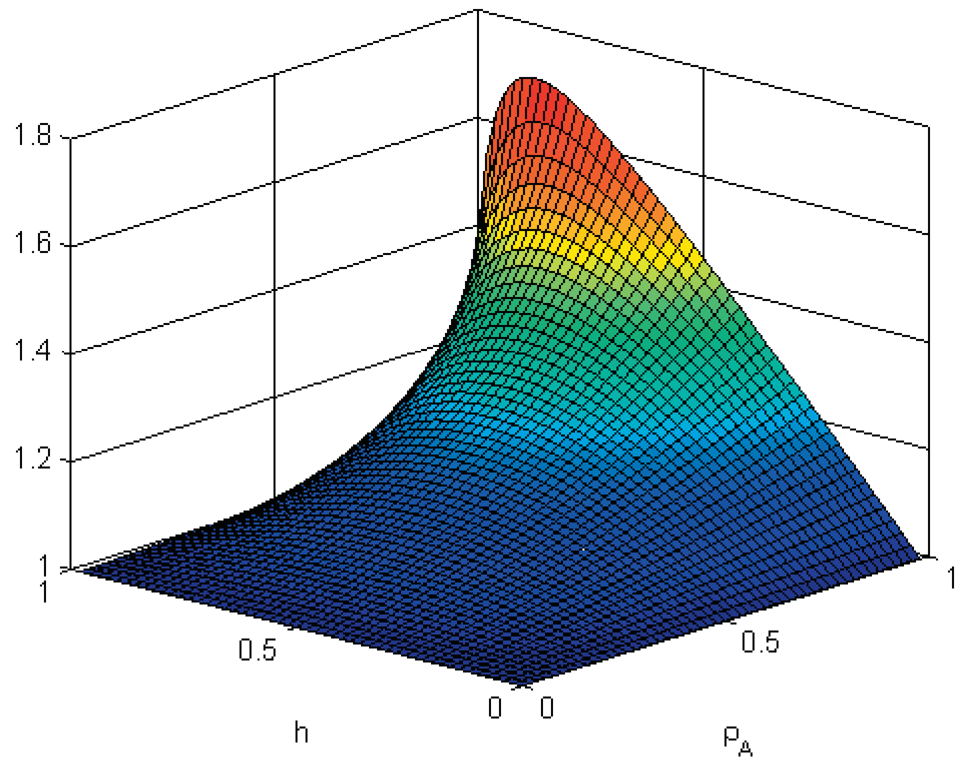
In this case, the market price of technological risk increases significantly compared to the plain vanilla case. This can be immediately appreciated when $\phi=\alpha=1$

$$\xi_\lambda\Big|_{\zeta=0,\frac{\phi}{\alpha}=1}=\frac{1-\rho_A\beta h^2}{1-\rho_A\beta h}\sigma_A \tag{38}$$

Under these assumptions, the market price of risk is always larger than in the plain-vanilla model, because $\frac{1-\rho_A\beta h^2}{1-\rho_A\beta h}>1$. Equation (38) is plotted as a function of ρ_A and h (and for $\beta=0.99$ and $\sigma_A=1$) in Figure 2. The figure shows that the price of risk is larger than in the plain-vanilla model only for a high persistence of technology shocks, as well as high, but not too high, habits. The price of technology risk is a nonlinear function of the habits parameter: it is initially increasing in h, but at some points it reaches a maximum and then it becomes decreasing in h. For the standard value of $\beta=0.99$, the maximum value of the price of technology risk occurs as $\rho_A\to1$ for $h\simeq0.9$; it is approximately equal to 1.7 for the $h=0.69$ value of our benchmark calibration. In the more general case of equation (37), using parameters values as in our benchmark calibration (but again $\sigma_A=1$ for ease of comparison), the price of technology risk would increase to slightly over 3.9.

Most of the intuition from this special case carries through to the more general model, which we analyse numerically.

Figure 2: The market price of technology risk as a function of ρ_A and h.



Note: special case where $\alpha = 1.0$ and $\phi = 1.0$.

8.4 The market price of technology risk in the full model

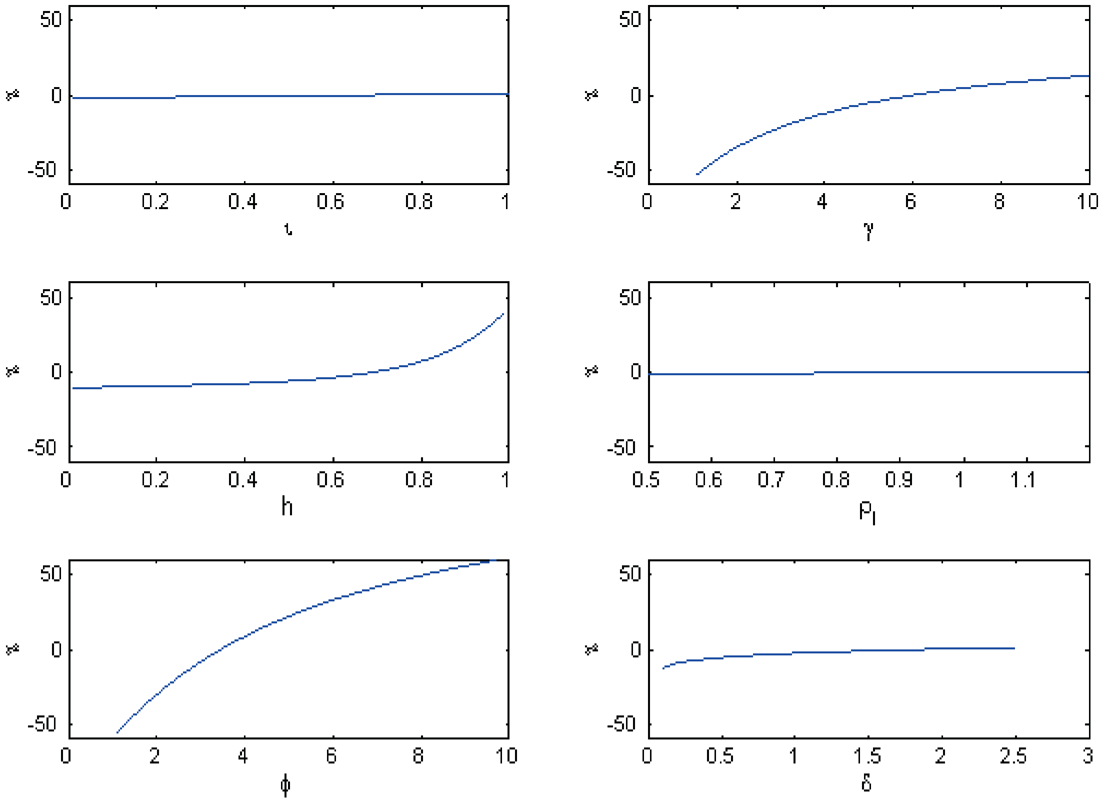
The calibration in Table 2 and Figure 1 generates a vector of market prices of risk, in annualised terms, approximately equal to 0.2 for technology shocks, negligible for inflation target shocks.[9] To compare this figure with the special cases analysed before, it can be noted that, for $\sigma_A = 1$, the market price of technology risk would be approximately equal to 3.1. This value is lower than the one which was obtained in the previous section within a model with habits and flexible prices (in the general case $\phi, \alpha \neq 1$). This suggests that habit formation is the main feature of our model which is necessary to generate the high excess holding period returns. The additional features of the model actually reduce the

[9] We compute the annualised market price of 1 unit of risk as $\sqrt{4}\xi$.

model's ability to generate high term premia. They are necessary, however, to prevent habit formation from producing undesired consequences on the variance of short-term rates.

We analyse the sensitivity of our benchmark result in Figure 3, where we perturb individual parameters from the benchmark specification and check the implications on the market prices of real risk. A similar exercise conducted on the market prices of inflation risk would show that these always remain negligible.

Figure 3: Sensitivity of the price of real risk to various parameters
(percentage changes from the baseline)



Note: all changes are with respect to the benchmark calibration exercise

Consistently with the results of the simplified model, Figure 3 highlights that the relative risk aversion coefficient, the habit persistence parameter and the elasticity of labour supply are the most important parameters affecting the market prices of real risk. Changes in the value of all these parameters can lead to variations in the market price of risk of up to ±50%. The inflation response parameter δ is also important, but it can only generate variations of about 10% in the market price of risk. Inflation indexation, which helped to solve the Backus, Gregory and Zin puzzle, and the degree of interest rate smoothing, which was important to generate the correct variance of yields, play a negligible role on the market price of risk.

8.5 A Fisher-type decomposition for holding premia

We conclude this section with a Fisher-type decomposition of holding premia.

Expected excess holding period return on nominal bonds can be written as

$$\widehat{hpr}_{t,n} - \widehat{i}_t = \left(\widehat{hpr}^{\text{real}}_{t,n} - \widehat{r}_t\right) + \left(\widehat{rp}_{t,n} - \widehat{rp}_{t,1}\right) + conv_{t,n} \tag{39}$$

where

$$\begin{aligned}
\widehat{hpr}^{\text{real}}_{t,n} - \widehat{r}_t &= \sigma^2 \lambda_x' \left(I - c_1^{n-1}\right) \eta\eta' \lambda_x \\
\widehat{rp}_{t,n} - \widehat{rp}_{t,1} &= \sigma \lambda_x' c_1^{n-1} \eta \xi_\pi - \sigma \pi_x' \left(I - c_1\right)^{-1} \left(I - c_1^{n-1}\right) c_1 \eta \xi_\lambda \\
conv_{t,n} &= -\sigma^2 \pi_x' c_1 \left(I - c_1\right)^{-1} \left(I - c_1^{n-1}\right) \eta\eta' \pi_x
\end{aligned}$$

Hence, nominal $xhpr$'s are equal to real $xhpr$'s plus two terms which arise because of inflation risk: an inflation risk premium and a convexity term.

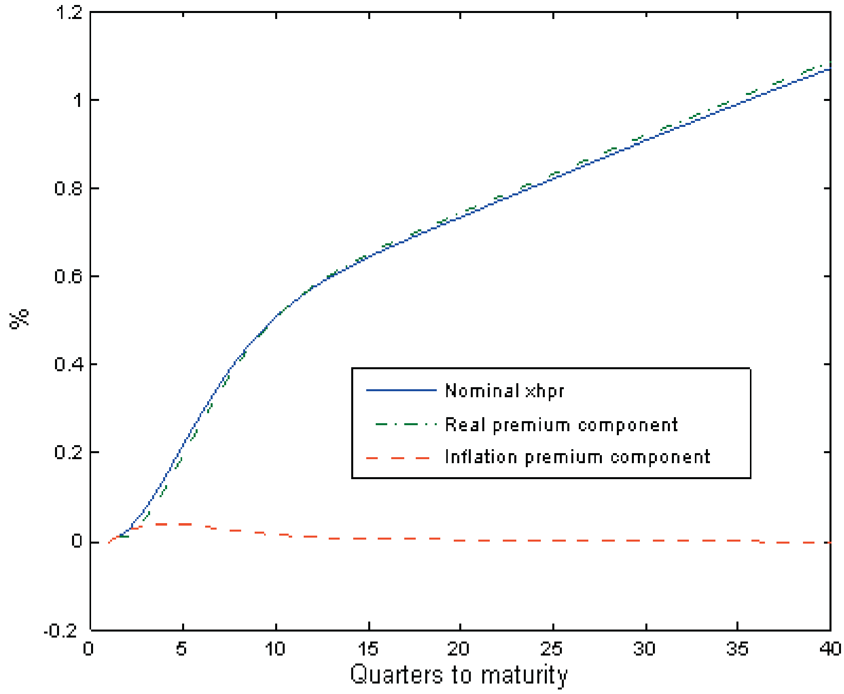
The excess inflation risk premium, in turn, includes two components. The first, $-\sigma\pi_x' \left(I - c_1\right)^{-1} \left(I - c_1^{n-1}\right) c_1 \eta \xi_\lambda$, is due to the risk of a surprisingly low real return on long term bonds because of the possibility of future inflation surprises. This risk is valued using the market price of real risk, ξ_λ, and it is such that positive premia are associated with a positive correlation between inflation and the marginal utility of consumption. The intuition is straightforward. Inflationary shocks are relevant to the household only to the extent that they have an impact on marginal utility. A positive covariance implies that an inflationary shock, which generates a surprise reduction of the real return on nominal bonds, produces, at the same time, an increase in the current marginal utility of consumption with respect to expected future marginal utility. In this sense, inflationary shocks have adverse effects and households try to ensure against their occurrence by requiring a risk premium on nominal bonds. The second component of the inflation risk premium, $\sigma\lambda_x' c_1^{n-1} \eta \xi_\pi$, represents instead the possibility that future surprises in the marginal utility of consumption will affect inflation over and above their direct effect on the real excess holding period return. This sort of risk is valued using the market price of inflation risk.

The convexity term, $conv_{t,n}$ is linked to the variance of inflation. When shocks tend to be positively serially correlated (formally, $c_1 (I - c_1)^{-1} (I - c_1^{n-1}) \eta\eta'$ is positive definite), the convexity term is always negative and tends to reduce the nominal $xhpr$.

Figure 4 shows the above decomposition for the benchmark case considered in Table 3. Clearly, the real $xhpr$ plays a vastly predominant role in shaping the nominal $xhpr$ at all maturities. The inflation risk premium is small around the 1-year maturity and virtually zero for other maturities. The convexity term is negligible across the whole maturity spectrum.

It should be emphasised here that our results regarding the size of inflation risk premia rely on a model where there is full credibility of monetary policy and where the inflation target, albeit time-varying, is well-anchored in the long run and perfectly understood and known by all agents. Relaxing these assumptions could obviously lead to different conclusions on inflation risk premia.

Figure 4: Decomposition of steady state excess holding period returns



Note: Parameter values as in the main calibration exercise: $\gamma = 2.6$. The horizontal axis gives the maturity in quarters.

9 Conclusions

This paper analyses the term structure implications of a relatively standard microfounded DSGE model with nominal rigidities. The model includes features typically embodied in DSGE models with sticky prices, such as inflation indexation, interest rate smoothing and habit persistence. Our interest in analyzing such a model originates in the fact that the literature has largely been unable to reconcile features of bond yield data with those of macro data within general equilibrium frameworks.

Using second-order approximate solutions to allow for a meaningful role for risk premia, our results show that DSGE models that incorporate the above-mentioned features are much closer to term structure data than previously thought. A relatively simple version of the model with only two persistent exogenous shocks can replicate quite well the sign and size of average excess holding period returns and the variance of yields across the term structure without unrealistic repercussions on the unconditional moments of macroeconomic variables.

While our approach based on second-order approximations allows us to derive model-consistent term premia endogenously, it can generate only time-invariant premia by construction. Hence, although we are able to match relatively well average features of term structure and macro data over a long period of time, the model is silent on any possible time-variation in risk premia. Given that the empirical finance literature has shown that risk premia do seem to vary over time (see e.g. Duffee, 2002 and Dai and Singleton, 2002), the ability of DSGE models to yield state dependent prices of risk that can match dynamics in observed data remains an important topic for future investigation. In order to address this issue, we must either assume shocks to be conditionally heteroskedastic (see for example Canova and Marrinan, 1996) or employ approximation methods for the model solution of order higher than two.

A Appendix

A.1 Calvo pricing

The optimisation problem for the firm under Calvo pricing is to maximise the stream of future profits based on a price which, with probability ζ, will not be re-optimised in the future, but simply indexed to an average of steady-state inflation, $\overline{\Pi}$, and lagged inflation, P_t/P_{t-1}. The problem can be stated as

$$\max_{P_t^i} \mathrm{E}_t \sum_{s=t}^{\infty} \zeta^{s-t} Q_{t,s} P_t^i \left(\overline{\Pi}^{s-t}\right)^{1-\iota} \left(\frac{P_{s-1}}{P_{t-1}}\right)^{\iota} Y_s^i \left[P_t^i \left(\overline{\Pi}^{s-t}\right)^{1-\iota} \left(\frac{P_{s-1}}{P_{t-1}}\right)^{\iota}\right]$$
$$-\mathrm{E}_t \sum_{s=t}^{\infty} \zeta^{s-t} Q_{t,s} TC_s \left(Y_s^i \left[P_t^i \left(\overline{\Pi}^{s-t}\right)^{1-\iota} \left(\frac{P_{s-1}}{P_{t-1}}\right)^{\iota}\right]\right)$$

where TC_s represents total costs and square brackets indicate that output at time $t+s$, Y_s^i, is a function of the price set at time t, namely $Y_s^i = \left(\frac{P_t^i\left(\overline{\Pi}^{s-t}\right)^{1-\iota}\left(\frac{P_{s-1}}{P_{t-1}}\right)^{\iota}}{P_s}\right)^{-\theta} Y_s$.

The FOCs can be written as

$$\mathrm{E}_t \sum_{s=t}^{\infty} \zeta^{s-t} Q_{t,s} \left((1-\theta)\left(\overline{\Pi}^{s-t}\right)^{1-\iota} \left(\frac{P_{s-1}}{P_{t-1}}\right)^{\iota} Y_s^i[\cdot] + \theta P_s mc_s \frac{Y_s^i[\cdot]}{P_t^i}\right) = 0$$

where marginal costs mc_s are given by

$$mc_s = \frac{\phi\chi}{\alpha}\frac{1}{\Lambda_s} A_s^{-\frac{\phi}{\alpha}} \left(\frac{P_t^i\left(\overline{\Pi}^{s-t}\right)^{1-\iota}\left(\frac{P_{s-1}}{P_{t-1}}\right)^{\iota}}{P_s}\right)^{-\theta\left(\frac{\phi}{\alpha}-1\right)} Y_s^{\frac{\phi}{\alpha}-1}$$

Substituting in this expression and rearranging terms, we obtain

$$\frac{P_t^i}{P_t} = \frac{\phi\chi}{\alpha} \frac{\mathrm{E}_t \sum_{s=t}^{\infty} \zeta^{s-t} Q_{t,s} \theta \frac{A_s^{-\frac{\phi}{\alpha}}}{\widetilde{\lambda}_s} \frac{P_s}{P_t} \left(\overline{\Pi}^{s-t}\right)^{-\theta\frac{\phi}{\alpha}(1-\iota)} \left(\frac{P_{s-1}}{P_{t-1}}\right)^{-\theta\frac{\phi}{\alpha}\iota} \left(\frac{P_t^i}{P_s}\right)^{-\theta\frac{\phi}{\alpha}} Y_s^{\frac{\phi}{\alpha}}}{\mathrm{E}_t \sum_{s=t}^{\infty} \zeta^{s-t} Q_{t,s} (\theta-1) \left(\overline{\Pi}^{s-t}\right)^{(1-\theta)(1-\iota)} \left(\frac{P_{s-1}}{P_{t-1}}\right)^{(1-\theta)\iota} \left(\frac{P_t^i}{P_s}\right)^{-\theta} Y_s}$$

Since all firms are identical in all respects but the time when they have the opportunity to change prices, the optimal decision they face at time t is the same in expected terms. They will therefore select the same price when they are given the opportunity, $P_t^i = P_t^*$, so that the above expression can be written as

$$\left(\frac{P_t^*}{P_t}\right)^{1-\theta\left(1-\frac{\phi}{\alpha}\right)} = \frac{\phi\chi\theta}{\alpha(\theta-1)} \frac{\mathrm{E}_t \sum_{s=t}^{\infty} \zeta^{s-t} Q_{t,s} \frac{A_s^{-\frac{\phi}{\alpha}}}{\widetilde{\lambda}_s} \left(\overline{\Pi}^{s-t}\right)^{-\theta\frac{\phi}{\alpha}(1-\iota)} \left(\frac{P_{s-1}}{P_{t-1}}\right)^{-\theta\frac{\phi}{\alpha}\iota} \left(\frac{P_t}{P_s}\right)^{-\theta\frac{\phi}{\alpha}-1} Y_s^{\frac{\phi}{\alpha}}}{\mathrm{E}_t \sum_{s=t}^{\infty} \zeta^{s-t} Q_{t,s} \left(\overline{\Pi}^{s-t}\right)^{(1-\theta)(1-\iota)} \left(\frac{P_{s-1}}{P_{t-1}}\right)^{(1-\theta)\iota} \left(\frac{P_t}{P_s}\right)^{-\theta} Y_s}$$

In order to approximate these relationships, we would like to get rid of the infinite sums. For this reason, we define

$$K_{2,t} \equiv \mathrm{E}_t \sum_{s=t}^{\infty} \zeta^{s-t} Q_{t,s} \frac{A_s^{-\frac{\phi}{\alpha}}}{\widetilde{\lambda}_s} \left(\overline{\Pi}^{s-t}\right)^{-\theta\frac{\phi}{\alpha}(1-\iota)} \left(\frac{P_{s-1}}{P_{t-1}}\right)^{-\theta\frac{\phi}{\alpha}\iota} \left(\frac{P_t}{P_s}\right)^{-\theta\frac{\phi}{\alpha}-1} Y_s^{\frac{\phi}{\alpha}}$$

so that

$$K_{2,t} = \frac{A_t^{-\frac{\phi}{\alpha}}}{\widetilde{\lambda}_t} Y_t^{\frac{\phi}{\alpha}} + \mathrm{E}_t \zeta \overline{\Pi}^{-\theta\frac{\phi}{\alpha}(1-\iota)} Q_{t,t+1} K_{2,t+1} \left(\frac{P_{t-1}}{P_t}\right)^{\theta\frac{\phi}{\alpha}\iota} \left(\frac{P_{t+1}}{P_t}\right)^{1+\theta\frac{\phi}{\alpha}}$$

and

$$K_{1,t} \equiv \mathrm{E}_t \sum_{s=t}^{\infty} \zeta^{s-t} Q_{t,s} \left(\overline{\Pi}^{s-t}\right)^{(1-\theta)(1-\iota)} \left(\frac{P_{s-1}}{P_{t-1}}\right)^{(1-\theta)\iota} \left(\frac{P_t}{P_s}\right)^{-\theta} Y_s$$

so that

$$K_{1,t} = Y_t + \mathrm{E}_t \zeta \overline{\Pi}^{(1-\theta)(1-\iota)} Q_{t,t+1} K_{1,t+1} \left(\frac{P_{t-1}}{P_t}\right)^{-(1-\theta)\iota} \left(\frac{P_{t+1}}{P_t}\right)^{\theta}$$

Hence, the first order conditions of the firms optimisation problem can be written as

$$\begin{aligned}
\left(\frac{P_t^*}{P_t}\right)^{1-\theta\left(1-\frac{\phi}{\alpha}\right)} &= \frac{\phi\chi\theta}{\alpha(\theta-1)} \frac{K_{2,t}}{K_{1,t}} \\
K_{2,t} &= \frac{A_t^{-\frac{\phi}{\alpha}}}{\widetilde{\lambda}_t} Y_t^{\frac{\phi}{\alpha}} + \mathrm{E}_t \zeta \overline{\Pi}^{-\theta\frac{\phi}{\alpha}(1-\iota)} Q_{t,t+1} K_{2,t+1} \Pi_t^{-\theta\frac{\phi}{\alpha}\iota} \Pi_{t+1}^{1+\theta\frac{\phi}{\alpha}} \\
K_{1,t} &= Y_t + \mathrm{E}_t \zeta \overline{\Pi}^{(1-\theta)(1-\iota)} Q_{t,t+1} K_{1,t+1} \Pi_t^{(1-\theta)\iota} \Pi_{t+1}^{\theta}
\end{aligned}$$

that are reproduced as equations (10) in the text.

Finally, note that the aggregate price can be written as

$$P_t = \left[(1-\zeta) P_t^{*1-\theta} + \zeta \left(P_{t-1} \overline{\Pi}^{1-\iota} \left(\frac{P_{t-1}}{P_{t-2}}\right)^{\iota}\right)^{1-\theta}\right]^{\frac{1}{1-\theta}}$$

so that

$$\frac{P_t^*}{P_t} = \left(\frac{1-\zeta\left(\overline{\Pi}^{1-\iota}\frac{\Pi_{t-1}^{\iota}}{\Pi_t}\right)^{1-\theta}}{(1-\zeta)}\right)^{\frac{1}{1-\theta}}$$

A.2 Steady state

From (10) and (11) we get

$$\begin{aligned}
K_1 &= Y + \zeta (\Pi)^{(1-\iota)(1-\theta)} \Pi^{\iota(1-\theta)} Q \Pi^{\theta} K_1 \\
K_1 &= \frac{Y}{1-\zeta\Pi Q} \qquad (40)
\end{aligned}$$

$$K_2 = Y^{\frac{\phi}{\alpha}}\Lambda^{-1}A^{-\frac{\phi}{\alpha}} + \zeta\Pi^{-\theta\frac{\phi}{\alpha}(1-\iota)}\Pi^{-\theta\iota\frac{\phi}{\alpha}}Q\Pi^{1+\theta\frac{\phi}{\alpha}}K_2$$

$$K_2 = \frac{Y^{\frac{\phi}{\alpha}}\Lambda^{-1}A^{-\frac{\phi}{\alpha}}}{1-\zeta\Pi Q} \tag{41}$$

Dividing these two expressions gives

$$\frac{K_2}{K_1} = Y^{\frac{\phi}{\alpha}-1}\Lambda^{-1}A^{-\frac{\phi}{\alpha}}.$$

By construction, $A = 1$, which hence simplifies the above to

$$\frac{K_2}{K_1} = Y^{\frac{\phi}{\alpha}-1}\Lambda^{-1}. \tag{42}$$

Next,

$$\frac{K_{2,t}}{K_{1,t}} = \frac{\theta-1}{\theta}\frac{\alpha}{\phi\chi} \tag{43}$$

Furthermore, from (6),

$$\Lambda = (Y-hY)^{-\gamma} - h\beta\,(Y-hY)^{-\gamma}$$

$$\Lambda = (1-h)^{-\gamma}\,(1-h\beta)\,Y^{-\gamma} \tag{44}$$

Combining (42), (43) and (44),

$$Y^{\frac{\phi}{\alpha}-1} = (1-h)^{-\gamma}\,(1-h\beta)\,Y^{-\gamma}\frac{\theta-1}{\theta}\frac{\alpha}{\phi\chi}$$

$$Y = \left(\frac{\theta-1}{\theta}\frac{\alpha}{\phi\chi}\frac{1-h\beta}{(1-h)^{\gamma}}\right)^{\frac{1}{\frac{\phi}{\alpha}+\gamma-1}} \tag{45}$$

(45) is our first result: it tells us how the steady state level of output relates to the structural parameters.

Since steady state output equals steady state natural output, the policy rule gives

$$I = \beta^{-1}\Pi^{\delta}$$

The Euler equation (7) gives us

$$\frac{1}{I} = \beta\Pi^{-1}$$

$$I = \beta^{-1}\Pi \tag{46}$$

Combining these two equations,

$$\Pi = \Pi^{\delta}$$

$$\Pi = 1,$$

and then (46) gives I. The definition of Q then gives

$$Q = \frac{\beta}{\Pi}.$$

Then (40) and (41) give us K_1 and K_2.

A.3 Model solution

The approximate solution of the model is computed following Schmitt-Grohé and Uribe (2004). First, we collect all first order conditions in a vector function F such that

$$F_t(x_t, \sigma) \equiv \mathrm{E}_t f(z_{t+1}, z_t, x_{t+1}, x_t) = 0$$

where x_t is the vector of (natural logarithms of the) predetermined variables and z_t is the vector of (natural logarithms of the) non-predetermined variables. More specifically, $x_t = [\pi_{t-1}, y_{t-1}, i_{t-1}, a_t, \pi_t^*]'$ and $z_t = [k_{1,t}, k_{2,t}, \pi_t, i_t, y_t, \lambda_t]'$. In F_t, σ denotes a scalar perturbation parameter, such that the law of motion of the exogenous state variables x_t^{exog} (where $x_t^{exog} = [a_t, \pi_t^*]'$) can be written as $x_{t+1}^{exog} = c_1^{exog} x_t^{exog} + \widetilde{\eta}\sigma\varepsilon_{t+1}$, for a known $n_\varepsilon \times n_\varepsilon$ matrix $\widetilde{\eta}$.

This system of equation is then solved nonlinearly at the deterministic steady state, namely where $x_{t+1} = x_t = \overline{x}$, $z_{t+1} = z_t = \overline{z}$ and $\sigma = 0$. Next, first and second order approximate solutions are derived recursively.

In the first order case, we postulate that the solution is affine, namely that $z_t = \overline{z} + z_x(x_t - \overline{x}) + z_\sigma \sigma$ and $x_{t+1} = \overline{x} + c_1(x_t - \overline{x}) + c_\sigma \sigma + \eta\sigma\varepsilon_{t+1}$ (where η is a $n_x \times n_\varepsilon$ matrix including zeros for the rows corresponding to endogenous predetermined variables and $\widetilde{\eta}$ for the rows corresponding to the exogenous states). When these expressions are used in the first order derivatives of $F_t(x_t, \sigma)$ with respect to x_t and σ evaluated at the deterministic steady state $(\overline{x}, 0)$, we obtain a system of linear equations which can be solved for the coefficients z_x, z_σ, c_1, c_σ. Since it is in general the case that $z_\sigma = c_\sigma = 0$, the first order solution can be written as (using a hat to denote deviations from the deterministic steady state) $\widehat{z}_t = z_x \widehat{x}_t$ and $\widehat{x}_{t+1} = c_1 \widehat{x}_t + \eta\sigma\varepsilon_{t+1}$ where z_x and c_1 are now known parameters.

The second order solution is obtained following exactly the same steps, except from the fact that second order derivatives of F_t need to be computed. The result can be written as

$$\widehat{z}_t = z_x\widehat{x}_t + \frac{1}{2}\begin{bmatrix} \widehat{x}_t' z_{xx}[1]\,\widehat{x}_t \\ \ldots \\ \widehat{x}_t' z_{xx}[n_z]\,\widehat{x}_t \end{bmatrix} + \frac{1}{2}z_{\sigma\sigma}\sigma^2$$

$$\widehat{x}_{t+1} = c_1\widehat{x}_t + \frac{1}{2}\begin{bmatrix} \widehat{x}_t' c_2[1]\,\widehat{x}_t \\ \ldots \\ \widehat{x}_t' c_2[n_x]\,\widehat{x}_t \end{bmatrix} + \frac{1}{2}c_0\sigma^2 + \eta\sigma\varepsilon_{t+1}$$

where $z_{xx}[i]$ and $c_2[i]$ are symmetric $n_x \times n_x$ matrices.

A.3.1 Special case: flexible prices and habit persistence with technology shocks

In this case (i.e. $\iota = \rho_I = \kappa = \zeta = 0$), the log-difference of the natural output equation is exactly

$$-\lambda_t - \frac{\phi}{\alpha}a_t + \left(\frac{\phi}{\alpha} - 1\right)y_t = 0$$

while a first order approximation of the marginal utility of consumption is

$$(1-\beta h)(1-h)\lambda_t = -\gamma\left[\left(1+\beta h^2\right)c_t - hc_{t-1} - \beta h \mathrm{E}_t c_{t+1}\right] \tag{47}$$

Combining these equations, we obtain

$$-\frac{\phi(1-\beta h)(1-h)}{\alpha\beta\gamma h}a_t = -\frac{\omega}{\beta\gamma h}y_t^n + \beta^{-1}y_{t-1}^n + \mathrm{E}_t y_{t+1}$$

where we define $\omega \equiv \gamma\left(1+\beta h^2\right) + (1-\beta h)(1-h)\left(\frac{\phi}{\alpha} - 1\right)$. This equation can be factorised as

$$-\frac{\phi(1-\beta h)(1-h)}{\alpha\beta\gamma h}a_t = (1-\tau_1 L)(1-\tau_2 L)\,\mathrm{E}_t y_{t+1}$$

where, using $(1-\tau_1 L)(1-\tau_2 L) = 1 - (\tau_1+\tau_2)L + \tau_1\tau_2 L^2$, we define

$$\tau_1\tau_2 = \beta^{-1}$$

$$\tau_1 + \tau_2 = \frac{\omega}{\beta\gamma h}$$

It follows that τ_2 solves

$$\beta^{-1} + \tau_2^2 - \frac{\omega}{\beta\gamma h}\tau_2 = 0$$

whose solutions are

$$\tau_2 = \frac{\frac{\omega}{\beta\gamma h} \pm \sqrt{\left(\frac{\omega}{\beta\gamma h}\right)^2 - 4\beta^{-1}}}{2}$$

The stable root is given by

$$\tau_2 = \frac{\omega - \sqrt{\omega^2 - 4\beta\gamma^2 h^2}}{2\beta\gamma h}$$

Since τ_1 is unstable, it has to be solved forward, which is why we take out $\tau_1 L$

$$-\frac{\phi\left(1-\beta h\right)\left(1-h\right)}{\alpha\beta\gamma h}a_t = \tau_1 L\left(\tau_1^{-1}L^{-1}-1\right)\left(1-\tau_2 L\right)\mathrm{E}_t y_{t+1}$$

so that

$$\left(1-\tau_2 L\right)y_t = \frac{\phi\left(1-\beta h\right)\left(1-h\right)}{\alpha\beta\gamma h\tau_1}\frac{1}{1-\tau_1^{-1}\rho_A}a_t$$

and we finally get

$$y_t = \tau_2 y_{t-1} + \tau_3 a_t \tag{48}$$

where $\tau_3 = \frac{\phi(1-\beta h)(1-h)}{\alpha\beta\gamma h(\tau_1-\rho_A)}$.

A.3.2 Special case: flexible prices with technology shocks

In the absence of habits (i.e. $h = \iota = \rho_I = \kappa = \zeta = 0$), the special case above boils down to simply

$$y_t = \frac{\frac{\phi}{\alpha}}{\frac{\phi}{\alpha} - 1 + \gamma}a_t$$

A.3.3 Special case: sticky-prices with technology shocks and without any other sources of endogenous persistence

In this case (i.e. $h = \iota = \rho_I = \kappa = 0$), the solution is more cumbersome. We use Mathematica code to show that

$$\begin{aligned} y_t &= \frac{\frac{\phi}{\alpha}}{\frac{\phi}{\alpha} - 1 + \gamma\,\Omega}a_t \\ \pi_t &= -\delta\frac{1-\rho_A}{\delta-\rho_A}\frac{\frac{\phi}{\alpha}}{\frac{\phi}{\alpha} - 1 + \gamma\,\Omega}a_t \end{aligned}$$

where

$$\Omega \equiv \frac{1}{\left(1-\zeta\right)\left(1-\beta\,\zeta\right)}\left[1+\zeta\left(\frac{\left(1-\rho_A\right)\left(1-\beta\,\rho_A\right)\theta}{\delta-\rho_A}\left(\frac{\phi}{\alpha}-1\right) - \frac{\delta-1}{\delta-\rho_A} - \beta\,\frac{\delta-\rho_A^2}{\delta-\rho_A} + \beta\,\zeta\right)\right]$$

Note that the term

$$\frac{\phi}{\alpha} - 1 + \gamma\,\Omega > 0$$

is always positive, since this implies (recall $\left(1-\zeta\right)\left(1-\beta\zeta\right) > 0$)

$$\left(\frac{\phi}{\alpha}-1\right)\left(\left(1-\zeta\right)\left(1-\beta\zeta\right) + \gamma\zeta\frac{\left(1-\rho_A\right)\left(1-\beta\rho_A\right)\theta}{\delta-\rho_A}\right) + \gamma\left(1+\beta\zeta^2\right) > \gamma\zeta\frac{\delta-1}{\delta-\rho_A} + \gamma\zeta\beta\frac{\delta-\rho_A^2}{\delta-\rho_A}$$

or

$$\left(\frac{\phi}{\alpha}-1\right)\left((1-\zeta)(1-\beta\zeta)+\gamma\zeta\frac{(1-\rho_A)(1-\beta\rho_A)\theta}{\delta-\rho_A}\right)+\gamma(1-\zeta)+\gamma\beta\zeta^2+\gamma\zeta\frac{1-\rho_A}{\delta-\rho_A}>\gamma\zeta\beta+\gamma\zeta\beta\frac{\rho_A(1-\rho_A)}{\delta-\rho_A}$$

or

$$\left(\frac{\phi}{\alpha}-1\right)\left((1-\zeta)(1-\beta\zeta)+\gamma\zeta\frac{(1-\rho_A)(1-\beta\rho_A)\theta}{\delta-\rho_A}\right)+\gamma(1-\zeta)(1-\zeta\beta)+\gamma\zeta\frac{(1-\rho_A)(1-\beta\rho_A)}{\delta-\rho_A}>0$$

which is always satisfied for $\rho_A<1$.

It is also the case that

$$\frac{\frac{\phi}{\alpha}}{\frac{\phi}{\alpha}-1+\gamma\,\Omega}<\frac{\frac{\phi}{\alpha}}{\frac{\phi}{\alpha}-1+\gamma}$$

namely that this coefficient is smaller in the sticky-price than in the flex-price case. This condition implies

$$\frac{\frac{\phi}{\alpha}}{\frac{\phi}{\alpha}-1+\gamma\left\{\frac{1}{(1-\zeta)(1-\beta\zeta)}\left[1+\zeta\left(\frac{(1-\rho_A)(1-\beta\rho_A)\theta}{\delta-\rho_A}\left(\frac{\phi}{\alpha}-1\right)-\frac{\delta-1}{\delta-\rho_A}-\beta\frac{\delta-\rho_A^2}{\delta-\rho_A}+\beta\zeta\right)\right]\right\}}<\frac{\frac{\phi}{\alpha}}{\frac{\phi}{\alpha}-1+\gamma}$$

or

$$(1-\zeta)(1-\beta\zeta)<1+\zeta\left(\frac{(1-\rho_A)(1-\beta\rho_A)\theta}{\delta-\rho_A}\left(\frac{\phi}{\alpha}-1\right)-\frac{\delta-1}{\delta-\rho_A}-\beta\frac{\delta-\rho_A^2}{\delta-\rho_A}+\beta\zeta\right)$$

or

$$(1-\zeta)(1-\beta\zeta)<1+\frac{\zeta}{\delta-\rho_A}\left((1-\rho_A)(1-\beta\rho_A)\theta\left(\frac{\phi}{\alpha}-1\right)-(\delta-1)-\beta\left(\delta-\rho_A^2\right)+\beta\zeta(\delta-\rho_A)\right)$$

or (since $\delta>\rho_A$)

$$(\delta-\rho_A)\left(1+\beta\zeta^2-\zeta(\beta+1)\right)<(\delta-\rho_A)\left(1+\beta\zeta^2\right)+\zeta(1-\rho_A)(1-\beta\rho_A)\theta\left(\frac{\phi}{\alpha}-1\right)-\zeta(\delta-1)-\zeta\beta\left(\delta-\rho_A^2\right)$$

or finally

$$0<\zeta(1-\rho_A)(1-\beta\rho_A)+\zeta(1-\rho_A)(1-\beta\rho_A)\theta\left(\frac{\phi}{\alpha}-1\right)$$

which is always satisfied.

For positive correlation coefficient ρ_A, the term $\frac{\delta(1-\rho_A)}{\delta-\rho_A}<1$ because $\rho_A(\delta-1)>0$. Also $\frac{\partial\frac{\delta(1-\rho_A)}{\delta-\rho_A}}{\partial\delta}=-\frac{\rho_A(1-\rho_A)}{(\rho_A-\delta)^2}<0$ and $\frac{\partial\frac{\delta(1-\rho_A)}{\delta-\rho_A}}{\partial\rho_A}=-\frac{\delta(\delta-1)}{(\rho_A-\delta)^2}<0$. Hence, the reduced-form inflation coefficient is always smaller than the output coefficient, and it is also decreasing in both ρ_A and δ.

A.4 Slope of the term structure

By definition,

$$B_{t,t+n} = E_t \left[Q_{t,t+n} \right]$$

and note that the non-stochastic steady state is such that $\overline{B} = \overline{Q}$ and $\ln \overline{B} = \ln \overline{Q}$. Now note that for any variable X, a second order approximation around the log-steady state is $X = \overline{x} \left(1 + \widehat{x} + \frac{1}{2}\widehat{x}^2\right)$, so that (simplifying the notation from $B_{t,t+n}$ to $B_{t,n}$)

$$\begin{aligned} \overline{b}\left(1 + \widehat{b}_{t,n} + \frac{1}{2}\widehat{b}_{t,n}^2\right) &= E_t\left[\overline{q}\left(1 + \widehat{q}_{t+n} + \frac{1}{2}\widehat{q}_{t+n}^2\right)\right] \\ &= \overline{q}E_t\left[1 + \widehat{q}_{t+n} + \frac{1}{2}\widehat{q}_{t+n}^2\right] \end{aligned}$$

or

$$\widehat{b}_{t,n} = E_t\left[\widehat{q}_{t+n} + \frac{1}{2}\widehat{q}_{t+n}^2\right] - \frac{1}{2}\widehat{b}_{t,n}^2$$

An expression for $\widehat{b}_{t,n}^2$ is needed, which can be obtained from its first order approximation. The latter can be denoted as

$$\overleftrightarrow{b}_{t,n} = E_t\left[\overleftrightarrow{q}_{t+n}\right]$$

so that

$$\begin{aligned} \widehat{b}_{t,n}^2 &= \left(E_t\left[\widehat{q}_{t+n}\right]\right)^2 \\ \widehat{b}_{t,n} &= E_t\left[\widehat{q}_{t+n}\right] + \frac{1}{2}E_t\left[\widehat{q}_{t+n}^2\right] - \frac{1}{2}\left(E_t\left[\widehat{q}_{t+n}\right]\right)^2 \end{aligned}$$

The price of a 1-period bond is therefore

$$\widehat{b}_{t,1} = -i_t = E_t\left[\widehat{q}_{t+1}\right] + \frac{1}{2}Var\left[\widehat{q}_{t+1}\right]$$

Now note that the log-stochastic discount factor is $\widehat{q}_{t,t+1} = \Delta\widehat{\lambda}_{t+1} - \widehat{\pi}_{t+1}$, so that $\widehat{q}_{t,t+n} = \widehat{\lambda}_{t+n} - \widehat{\lambda}_{t+1} - \sum_{i=1}^n \widehat{\pi}_{t+i} \equiv \Delta^n\widehat{\lambda}_{t+n} - \sum_{i=1}^n \widehat{\pi}_{t+i}$ and

$$\widehat{b}_{t,n} = \mathrm{E}_t\left[\Delta^n\widehat{\lambda}_{t+n}\right] - \sum_{i=1}^n \mathrm{E}_t\left[\widehat{\pi}_{t+i}\right] + \frac{1}{2}\mathrm{Var}_t\left[\Delta^n\widehat{\lambda}_{t+n} - \sum_{i=1}^n \widehat{\pi}_{t+i}\right]$$

Since $\widehat{ytm}_{t,n} = -\left(1/n\right)\widehat{b}_{t,n}$, we also obtain

$$\begin{aligned} \widehat{ytm}_{t,n} &= -\frac{\mathrm{E}_t\left[\Delta^n\widehat{\lambda}_{t+n}\right]}{n} + \frac{\sum_{i=1}^n \mathrm{E}_t\left[\widehat{\pi}_{t+i}\right]}{n} - \frac{1}{2}\frac{1}{n}\mathrm{Var}_t\left[\Delta^n\widehat{\lambda}_{t+n} - \sum_{i=1}^n \widehat{\pi}_{t+i}\right] \\ &= -\frac{\mathrm{E}_t\left[\Delta^n\widehat{\lambda}_{t+n}\right]}{n} + \frac{\sum_{i=1}^n \mathrm{E}_t\left[\widehat{\pi}_{t+i}\right]}{n} - \frac{1}{2}\frac{\mathrm{Var}_t\left[\Delta^n\widehat{\lambda}_{t+n}\right]}{n} \\ &\quad -\frac{1}{2}\frac{\mathrm{Var}_t\left[\sum_{i=1}^n \widehat{\pi}_{t+i}\right]}{n} + \frac{\mathrm{Cov}_t\left[\sum_{i=1}^n \widehat{\pi}_{t+i}, \Delta^n\widehat{\lambda}_{t+n}\right]}{n} \end{aligned}$$

and note that

$$\widehat{i}_t = \widehat{ytm}_{t,1} = -\mathrm{E}_t\left[\Delta\widehat{\lambda}_{t+1}\right] + \mathrm{E}_t\left[\widehat{\pi}_{t+1}\right] - \frac{1}{2}\mathrm{Var}_t\left[\Delta\widehat{\lambda}_{t+1}\right] - \frac{1}{2}\mathrm{Var}_t\left[\widehat{\pi}_{t+1}\right] + \mathrm{Cov}_t\left[\widehat{\pi}_{t+1}, \Delta\widehat{\lambda}_{t+1}\right]$$

It follows that

$$\begin{aligned} \widehat{ytm}_{t,n} - \widehat{i}_t &= -\left(\frac{1}{n}\mathrm{E}_t\left[\Delta^n\widehat{\lambda}_{t+n}\right] - \mathrm{E}_t\left[\Delta\widehat{\lambda}_{t+1}\right]\right) - \frac{1}{2}\left(\frac{1}{n}\mathrm{Var}_t\left[\Delta^n\widehat{\lambda}_{t+n}\right] - \mathrm{Var}_t\left[\Delta\widehat{\lambda}_{t+1}\right]\right) \\ &+ \left(\frac{\sum_{i=1}^{n}\mathrm{E}_t\left[\widehat{\pi}_{t+i}\right]}{n} - \mathrm{E}_t\left[\pi_{t+1}\right]\right) - \frac{1}{2}\left(\frac{1}{n}\mathrm{Var}_t\left[\sum_{i=1}^{n}\widehat{\pi}_{t+i}\right] - \mathrm{Var}_t\left[\widehat{\pi}_{t+1}\right]\right) \\ &+ \left(\frac{1}{n}\mathrm{Cov}_t\left[\sum_{i=1}^{n}\widehat{\pi}_{t+i}, \Delta^n\widehat{\lambda}_{t+n}\right] - \mathrm{Cov}_t\left[\widehat{\pi}_{t+1}, \Delta\widehat{\lambda}_{t+1}\right]\right) \end{aligned}$$

The unconditional slope is therefore

$$\begin{aligned} \mathrm{E}\left[\widehat{ytm}_{t,n} - \widehat{i}_t\right] &= -\frac{1}{2}\left(\frac{\mathrm{E}\left[\mathrm{Var}_t\left[\Delta^n\widehat{\lambda}_{t+n}\right]\right]}{n} - \mathrm{E}\left[\mathrm{Var}_t\left[\Delta\widehat{\lambda}_{t+1}\right]\right]\right) \\ &\quad -\frac{1}{2}\left(\frac{\mathrm{E}\left[\mathrm{Var}_t\left[\sum_{i=1}^{n}\widehat{\pi}_{t+i}\right]\right]}{n} - \mathrm{E}\left[\mathrm{Var}_t\left[\widehat{\pi}_{t+1}\right]\right]\right) \\ &\quad + \frac{\mathrm{E}\left[\mathrm{Cov}_t\left[\sum_{i=1}^{n}\widehat{\pi}_{t+i}, \Delta^n\widehat{\lambda}_{t+n}\right]\right]}{n} - \mathrm{E}\left[\mathrm{Cov}_t\left[\widehat{\pi}_{t+1}, \Delta\widehat{\lambda}_{t+1}\right]\right] \end{aligned}$$

Real returns can be obtained similarly as

$$\widehat{ytm}_{t,n}^{\mathrm{real}} = -\frac{\mathrm{E}_t\left[\Delta^n\widehat{\lambda}_{t+n}\right]}{n} - \frac{1}{2}\frac{\mathrm{Var}_t\left[\Delta^n\widehat{\lambda}_{t+n}\right]}{n}$$

where again note that

$$\widehat{r}_t = \widehat{ytm}_{t,1}^{\mathrm{real}} = -\mathrm{E}_t\left[\Delta\widehat{\lambda}_{t+1}\right] - \frac{1}{2}\mathrm{Var}_t\left[\Delta\widehat{\lambda}_{t+1}\right]$$

Hence

$$\widehat{ytm}_{t,n}^{\mathrm{real}} - \widehat{r}_t = -\left(\frac{1}{n}\mathrm{E}_t\left[\Delta^n\widehat{\lambda}_{t+n}\right] - \mathrm{E}_t\left[\Delta\widehat{\lambda}_{t+1}\right]\right) - \frac{1}{2}\left(\frac{1}{n}\mathrm{Var}_t\left[\Delta^n\widehat{\lambda}_{t+n}\right] - \mathrm{Var}_t\left[\Delta\widehat{\lambda}_{t+1}\right]\right)$$

and

$$\mathrm{E}\left[\widehat{ytm}_{n}^{\mathrm{real}} - \widehat{r}\right] = -\frac{1}{2}\left(\frac{\mathrm{E}\left[\mathrm{Var}_t\left[\Delta^n\widehat{\lambda}_{t+n}\right]\right]}{n} - \mathrm{E}\left[\mathrm{Var}_t\left[\Delta\widehat{\lambda}_{t+1}\right]\right]\right)$$

In the special case with CRRA utility, $\Delta^n\widehat{\lambda}_n = -\gamma\Delta^n\widehat{y}_n$.

A.5 Solution for bond prices

Recall that

$$
\begin{aligned}
\widehat{x}_{t+1} &= c_1\widehat{x}_t + \frac{1}{2}\begin{bmatrix} \cdots \\ \widehat{x}_t' c_{i2}\widehat{x}_t \\ \cdots \end{bmatrix} + \frac{1}{2}c_0\sigma^2 + \eta\sigma\varepsilon_{t+1} \\
\widehat{\lambda}_t &= \lambda_x'\widehat{x}_t + \frac{1}{2}\widehat{x}_t'\lambda_{xx}\widehat{x}_t + \frac{1}{2}\lambda_{\sigma\sigma}\sigma^2 \\
\widehat{\pi}_t &= \pi_x'\widehat{x}_t + \frac{1}{2}\widehat{x}_t'\pi_{xx}\widehat{x}_t + \frac{1}{2}\pi_{\sigma\sigma}\sigma^2
\end{aligned}
$$

and stack the state variables to write

$$
\widehat{x}_{t+1} = c_1\widehat{x}_t + \frac{1}{2}\begin{bmatrix} \cdots \\ \widehat{x}_t' c_{i2}\widehat{x}_t \\ \cdots \end{bmatrix} + \frac{1}{2}\sigma^2 c_0 + \sigma\eta\varepsilon_{t+1}
$$

where now c_1 is a $n_x \times n_x$ matrix, c_0 is a $n_x \times 1$ vector, c_{i2} are n_x matrices of dimension $n_x \times n_x$ each, and finally η is a $n_x \times n_\varepsilon$ matrix.

A.5.1 1-period bonds

Even if this is known from the solution for the short term rate, it is instructive to derive first the price of a 1-period bond. For this purpose, note first that a second order approximation to the stochastic discount factor is

$$
\widehat{q}_{t,t+1} = \left(\lambda_x' - \pi_x'\right)\widehat{x}_{t+1} + \frac{1}{2}\widehat{x}_{t+1}'\left(\lambda_{xx} - \pi_{xx}\right)\widehat{x}_{t+1} - \lambda_x'\widehat{x}_t - \frac{1}{2}\widehat{x}_t'\lambda_{xx}\widehat{x}_t - \frac{1}{2}\pi_{\sigma\sigma}\sigma^2
$$

or

$$
\begin{aligned}
\widehat{q}_{t,t+1} = {} & \left(\left(\lambda_x' - \pi_x'\right)c_1 - \lambda_x'\right)\widehat{x}_t + \frac{1}{2}\left(\lambda_x' - \pi_x'\right)\begin{bmatrix} \cdots \\ \widehat{x}_t' c_{i2}\widehat{x}_t \\ \cdots \end{bmatrix} \\
& + \frac{1}{2}\widehat{x}_t' c_1'\left(\lambda_{xx} - \pi_{xx}\right)c_1\widehat{x}_t - \frac{1}{2}\widehat{x}_t'\lambda_{xx}\widehat{x}_t \\
& + \frac{1}{2}\sigma^2\left(\lambda_x' - \pi_x'\right)c_0 - \frac{1}{2}\pi_{\sigma\sigma}\sigma^2 \\
& + \sigma\left(\lambda_x' - \pi_x'\right)\eta\varepsilon_{t+1} + \frac{1}{2}\sigma\widehat{x}_t' c_1'\left(\lambda_{xx} - \pi_{xx}\right)\eta\varepsilon_{t+1} \\
& + \frac{1}{2}\sigma\varepsilon_{t+1}'\eta'\left(\lambda_{xx} - \pi_{xx}\right)c_1\widehat{x}_t + \frac{1}{2}\sigma^2\varepsilon_{t+1}'\eta'\left(\lambda_{xx} - \pi_{xx}\right)\eta\varepsilon_{t+1}
\end{aligned}
$$

Now recall that the price of a 1-period bond is

$$
\widehat{b}_{t,1} = -i_t = \mathrm{E}_t\left[\widehat{q}_{t+1}\right] + \frac{1}{2}\left(\mathrm{E}_t\left[\widehat{q}_{t+1}^2\right] - \left(\mathrm{E}_t\left[\widehat{q}_{t+1}\right]\right)^2\right)
$$

for which we need

$$\mathrm{E}_t\left[\widehat{q}_{t+1}^2\right]-\left(\mathrm{E}_t\left[\widehat{q}_{t+1}\right]\right)^2=\sigma^2\left(\lambda_x'-\pi_x'\right)\eta\eta'\left(\lambda_x'-\pi_x'\right)'$$

and

$$\begin{aligned}\mathrm{E}_t\left[\widehat{q}_{t,t+1}\right] &= \left(\lambda_x'-\pi_x'\right)c_1\widehat{x}_t+\frac{1}{2}\left(\lambda_x'-\pi_x'\right)\begin{bmatrix}\cdots\\ \widehat{x}_t'c_{i2}\widehat{x}_t\\ \cdots\end{bmatrix}+\frac{1}{2}\sigma^2\left(\lambda_x'-\pi_x'\right)c_0\\ &+\frac{1}{2}\widehat{x}_t'c_1'\left(\lambda_{xx}-\pi_{xx}\right)c_1\widehat{x}_t+\frac{1}{2}\sigma^2\mathrm{E}_t\left[\varepsilon_{t+1}'\eta'\left(\lambda_{xx}-\pi_{xx}\right)\eta\varepsilon_{t+1}\right]\\ &-\lambda_x'\widehat{x}_t-\frac{1}{2}\widehat{x}_t'\lambda_{xx}\widehat{x}_t-\frac{1}{2}\pi_{\sigma\sigma}\sigma^2\end{aligned}$$

Now note that, for any matrix A and vector x,

$$\begin{aligned}\mathrm{E}\left[x'Ax\right] &= \mathrm{E}\left[\mathrm{vec}\left(x'Ax\right)\right]\\ &= \mathrm{E}\left[x'\otimes x'\right]\mathrm{vec}\left(A\right)\\ &= \left(\mathrm{vec}\left(\mathrm{E}\left[xx'\right]\right)\right)'\mathrm{vec}\left(A\right)\end{aligned}$$

where the vec operator transforms a matrix into a vector by stacking its columns. It follows that

$$\begin{aligned}\mathrm{E}_t\left[\varepsilon_{t+1}'\eta'\left(\lambda_{xx}-\pi_{xx}\right)\eta\varepsilon_{t+1}\right] &= \left(\mathrm{vec}\left(\mathrm{I}\right)\right)\mathrm{vec}\left(\eta'\left(\lambda_{xx}-\pi_{xx}\right)\eta\right)\\ &= \mathrm{tr}\left(\eta'\left(\lambda_{xx}-\pi_{xx}\right)\eta\right)\end{aligned}$$

where tr represents the trace, i.e. the sum of the diagonal elements of a matrix.

Hence,

$$\begin{aligned}\widehat{b}_{t,1} &= \left(\left(\lambda_x'-\pi_x'\right)c_1-\lambda_x'\right)\widehat{x}_t+\frac{1}{2}\sigma^2\left(\left(\lambda_x'-\pi_x'\right)c_0-\pi_{\sigma\sigma}\right)\\ &+\frac{1}{2}\sigma^2\mathrm{tr}\left(\eta'\left(\lambda_{xx}-\pi_{xx}\right)\eta\right)+\frac{1}{2}\sigma^2\left(\lambda_x'-\pi_x'\right)\eta\eta'\left(\lambda_x'-\pi_x'\right)'\\ &+\frac{1}{2}\widehat{x}_t'\left(c_1'\left(\lambda_{xx}-\pi_{xx}\right)c_1-\lambda_{xx}\right)\widehat{x}_t+\frac{1}{2}\left(\lambda_x'-\pi_x'\right)\begin{bmatrix}\cdots\\ \widehat{x}_t'c_{i2}\widehat{x}_t\\ \cdots\end{bmatrix}\end{aligned}$$

Finally, note that

$$\begin{aligned}\underset{1\times n_x}{\left(\lambda_x'-\pi_x'\right)}\begin{bmatrix}\cdots\\ \widehat{x}_t'c_{i2}\widehat{x}_t\\ \cdots\end{bmatrix} &= \sum_{i=1}^{n_x}\underset{1\times n_x}{\left(\lambda_x'\left[i\right]-\pi_x'\left[i\right]\right)}\widehat{x}_t'c_2\left[i\right]\widehat{x}_t\\ &= \sum_{i=1}^{n_x}\widehat{x}_t'\left(\lambda_x'\left[i\right]-\pi_x'\left[i\right]\right)c_2\left[i\right]\widehat{x}_t\\ &= \widehat{x}_t'\left(\sum_{i=1}^{n_x}\left(\lambda_x'\left[i\right]-\pi_x'\left[i\right]\right)c_2\left[i\right]\right)\widehat{x}_t\end{aligned}$$

where the second equality follows from the fact that $\lambda_x'[i]$ and $\pi_x'[i]$ are scalar elements of λ_x' and π_x', respectively. We can therefore rewrite the 1-period bond as

$$\widehat{b}_{t,1} = B_{1,x}'\widehat{x}_t + \frac{1}{2}\sigma^2 B_{1,\sigma\sigma} + \frac{1}{2}\widehat{x}_t' B_{1,xx}\widehat{x}_t$$

where

$$\begin{aligned}
B_{1,x}' &\equiv \left(\lambda_x' - \pi_x'\right) c_1 - \lambda_x' \\
B_{1,\sigma\sigma} &\equiv \left(\lambda_x' - \pi_x'\right) c_0 - \pi_{\sigma\sigma} + \operatorname{tr}\left(\eta' \left(\lambda_{xx} - \pi_{xx}\right)\eta\right) + \left(\lambda_x' - \pi_x'\right)\eta\eta'\left(\lambda_x' - \pi_x'\right)' \\
B_{1,xx} &\equiv c_1' \left(\lambda_{xx} - \pi_{xx}\right) c_1 - \lambda_{xx} + \sum_{i=1}^{n_x}\left(\lambda_x'[i] - \pi_x'[i]\right) c_2[i]
\end{aligned}$$

Note also that, by construction, $\mathrm{E}\left[\widehat{b}_1\right] = -\mathrm{E}\left[\widehat{i}\right]$, so $B_{1,x} = -\widehat{i}_x$, $B_{1,xx} = -\widehat{i}_{xx}$ and $B_{1,\sigma\sigma} = -\widehat{i}_{\sigma\sigma}$. Note that this definition also allows us to rewrite $\widehat{q}_{t,t+1}$ as

$$\begin{aligned}
\widehat{q}_{t,t+1} = {} & B_{1,x}'\widehat{x}_t + \frac{1}{2}\widehat{x}_t' B_{1,xx}\widehat{x}_t + \frac{1}{2}\sigma^2\left(\left(\lambda_x' - \pi_x'\right) c_0 - \pi_{\sigma\sigma}\right) \\
& + \sigma\left(\lambda_x' - \pi_x'\right)\eta\varepsilon_{t+1} + \sigma\widehat{x}_t' c_1' \left(\lambda_{xx} - \pi_{xx}\right)\eta\varepsilon_{t+1} + \frac{1}{2}\sigma^2\varepsilon_{t+1}'\eta'\left(\lambda_{xx} - \pi_{xx}\right)\eta\varepsilon_{t+1}
\end{aligned}$$

A.5.2 2-period bonds

2-period bond prices can be written as (up to a second order approximation)

$$\widehat{b}_{t,2} = \widehat{b}_{t,1} + \mathrm{E}_t\left[\widehat{b}_{t+1,1}\right] + \frac{1}{2}\mathrm{Var}_t\left[\widehat{b}_{t+1,1}\right] + \mathrm{Cov}_t\left[\widehat{q}_{t+1}, \widehat{b}_{t+1,1}\right]$$

Based on 1-period prices, we can derive

$$\begin{aligned}
\mathrm{E}_t\left[\widehat{b}_{t+1,1}\right] = {} & B_{1,x}' c_1 \widehat{x}_t + \frac{1}{2} B_{1,x}' \begin{bmatrix} \cdots \\ \widehat{x}_t' c_{i2}\widehat{x}_t \\ \cdots \end{bmatrix} + \frac{1}{2}\widehat{x}_t' c_1' B_{1,xx} c_1 \widehat{x}_t \\
& + \frac{1}{2} B_{1,x}' c_0 \sigma^2 + \frac{1}{2}\sigma^2 B_{1,\sigma\sigma} + \frac{1}{2}\sigma^2 \operatorname{tr}\left[\eta' B_{1,xx}\eta\right]
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{E}_t\left[\widehat{b}_{t+1,1}\widehat{b}_{t+1,1}'\right] - \mathrm{E}_t\left[\widehat{b}_{t+1,1}'\right]\mathrm{E}_t\left[\widehat{b}_{t+1,1}\right] &= \sigma^2 B_{1,x}'\eta\eta' B_{1,x} \\
\mathrm{E}_t\left[\widehat{b}_{t+1,1}\widehat{q}_{t+1}'\right] - \mathrm{E}_t\left[\widehat{b}_{t+1,1}\right]\mathrm{E}_t\left[\widehat{q}_{t+1}'\right] &= B_{1,x}'\eta\eta'\left(\lambda_x - \pi_x\right)
\end{aligned}$$

It follows that

$$\widehat{b}_{t,2} = B_{2,x}'\widehat{x}_t + \frac{1}{2}\widehat{x}_t' B_{2,xx}\widehat{x}_t + \frac{1}{2}\sigma^2 B_{2,\sigma\sigma}$$

where

$$\begin{aligned} B'_{2,x} &= B'_{1,x}\left(I + c_1\right) \\ B_{2,xx} &= B_{1,xx} + c'_1 B_{1,xx} c_1 + \sum_{i=1}^{n_x} B'_{1,x}\left[i\right] c_2\left[i\right] \\ B_{2,\sigma\sigma} &= 2B_{1,\sigma\sigma} + B'_{1,x} c_0 + \operatorname{tr}\left(\eta' B_{1,xx}\eta\right) + B'_{1,x}\eta\eta' B_{1,x} + 2B'_{1,x}\eta\eta'\left(\lambda_x - \pi_x\right) \end{aligned}$$

A.5.3 n-period bonds

Using the same procedure, we find that n-period bond prices can be written as

$$\widehat{b}_{t,n} = B'_{n,x}\widehat{x}_t + \frac{1}{2}\widehat{x}'_t B_{n,xx}\widehat{x}_t + \frac{1}{2}\sigma^2 B_{n,\sigma\sigma}$$

where for $n > 1$

$$\begin{aligned} B'_{n,x} &= B'_{1,x} + B'_{n-1,x} c_1 \\ B_{n,xx} &= B_{1,xx} + c'_1 B_{n-1,xx} c_1 + \sum_{i=1}^{n_x} B'_{n-1,x}\left[i\right] c_2\left[i\right] \\ B_{n,\sigma\sigma} &= B_{1,\sigma\sigma} + B_{n-1,\sigma\sigma} + B'_{n-1,x} c_0 + \operatorname{tr}\left(\eta' B_{n-1,xx}\eta\right) \\ & \quad + B'_{n-1,x}\eta\eta' B_{n-1,x} + 2B'_{n-1,x}\eta\eta'\left(\lambda_x - \pi_x\right) \end{aligned}$$

Note that the first order term $B'_{n,x}$ can be solved explicitly as

$$\begin{aligned} B'_{n,x} &= B'_{1,x} + B'_{n-1,x} c_1 \\ &= B'_{1,x}\sum_{j=0}^{n-1} c_1^j \\ &= B'_{1,x}\left(I - c_1\right)^{-1}\left(I - c_1^n\right) \end{aligned}$$

A.5.4 Unconditional moments of n-period bonds

The unconditional mean of an n-period bond is

$$\begin{aligned} \mathrm{E}\left[\widehat{b}_{t,n}\right] &= B'_{n,x}\mathrm{E}\left[\widehat{x}\right] + \frac{1}{2}\mathrm{E}\left[\widehat{x}' B_{n,xx}\widehat{x}\right] + \frac{1}{2}\sigma^2 B_{n,\sigma\sigma} \\ &= B'_{n,x}\mathrm{E}\left[\widehat{x}\right] + \frac{1}{2}\left(\operatorname{vec}\left(\mathrm{E}\left[xx'\right]\right)\right)'\operatorname{vec}\left(B_{n,xx}\right) + \frac{1}{2}\sigma^2 B_{n,\sigma\sigma} \end{aligned}$$

and the unconditional variance

$$\begin{aligned} \operatorname{Var}\left[\widehat{b}_{t,n}\right] &= \mathrm{E}\left[\left(B'_{1,x}\left(I - c_1\right)^{-1}\left(I - c_1^n\right)\widehat{x}_t\right)\left(B'_{1,x}\left(I - c_1\right)^{-1}\left(I - c_1^n\right)\widehat{x}_t\right)'\right] \\ &= \mathrm{E}\left[B'_{1,x}\left(I - c_1\right)^{-1}\left(I - c_1^n\right)\widehat{x}_t\widehat{x}'_t\left(I - c_1^n\right)'\left(\left(I - c_1\right)^{-1}\right)' B_{1,x}\right] \\ &= B'_{1,x}\left(I - c_1\right)^{-1}\left(I - c_1^n\right)\mathrm{E}\left[xx'\right]\left(I - c_1^n\right)'\left(\left(I - c_1\right)^{-1}\right)' B_{1,x} \end{aligned}$$

Incidentally, note that for real bonds $B'^{\text{real}}_{1,x} \equiv -\lambda'_x (I - c_1)$, so that

$$\text{Var}\left[\widehat{b}^{\text{real}}_{t,n}\right] = \lambda'_x (I - c_1^n) \,\text{E}\left[xx'\right] (I - c_1^n)' \,\lambda_x$$

For nominal yields, we obtain

$$\text{Var}\left[\widehat{ytm}_n\right] = \frac{1}{n^2} B'_{1,x} (I - c_1)^{-1} (I - c_1^n) \,\text{E}\left[xx'\right] (I - c_1^n)' \left((I - c_1)^{-1}\right)' B_{1,x}$$

Finally note the ratio

$$\frac{\text{Var}\left[\widehat{ytm}_n\right]}{\text{Var}\left[\widehat{i}\right]} = \frac{1}{n^2} \frac{B'_{1,x} (I - c_1)^{-1} (I - c_1^n) \,\text{E}\left[xx'\right] (I - c_1^n)' \left((I - c_1)^{-1}\right)' B_{1,x}}{B'_{1,x} \text{E}\left[xx'\right] B_{1,x}}$$

which in the scalar case, redefining c_1 as ρ, becomes

$$\frac{\text{Var}\left[\widehat{ytm}_n\right]}{\text{Var}\left[\widehat{i}\right]} = \left(\frac{1}{n}\frac{1 - \rho^n}{1 - \rho}\right)^2$$

A.6 Expected holding period return in the flex-price no-habit case

The model solution in the flex-price no-habit case is

$$Y_t = \left(\mu^{-1} \frac{\alpha}{\chi\phi}\right)^{\frac{\alpha}{\phi + \alpha(\gamma - 1)}} A_t^{\frac{\phi}{\phi + \alpha(\gamma - 1)}}$$

while bond prices are

$$N_{n,t} = \beta^n (Y_t)^{\gamma} \,\text{E}_t \left[(Y_{t+n})^{-\gamma}\right]$$

Substituting out output and evaluating expectations

$$N_{n,t} = \beta^n A_t^{\frac{\gamma\phi(1 - \rho_A^n)}{\gamma\alpha - \alpha + \phi}} e^{\frac{1}{2}\frac{1 - \rho_A^{2n}}{1 - \rho_A^2}\left(\frac{\gamma\phi}{\phi - \alpha(1 - \gamma)}\right)^2 \sigma_\varepsilon^2}$$

which characterises the whole term-structure in closed form. Note that, if we focus on continuously compounded returns defined as $\bar{i}_{n,t} = -\frac{1}{n} \ln N_{n,t}$, we obtain

$$\widehat{ytm}_{n,t} = a_n + b_n \ln A_t$$

where

$$a_n \equiv -\ln\beta - \frac{1}{2n}\frac{1 - \rho_A^{2n}}{1 - \rho_A^2}\left(\frac{\gamma\phi}{\phi + \alpha(\gamma - 1)}\right)^2 \sigma_\varepsilon^2$$

$$b_n \equiv -\frac{1}{n}\frac{\gamma\phi(1 - \rho_A^n)}{\phi + \alpha(\gamma - 1)}$$

Finally, note that the expected holding period return is

$$\begin{aligned} HPR_{n,t} &= E_t\left[\frac{N_{t+1}^{n-1}}{N_t^n}\right] \\ &= \frac{e^{\frac{1}{2}\left(\frac{\gamma\phi}{\gamma\alpha-\alpha+\phi}\right)^2(1-2\rho_A^{n-1})\sigma_\varepsilon^2}}{\beta} A_t^{-\frac{\gamma\phi(1-\rho_A)}{\gamma\alpha-\alpha+\phi}} \end{aligned}$$

so that

$$\begin{aligned} hpr_{t,n} &\equiv \ln HPR_{t,n} \\ &= \ln \frac{e^{\frac{1}{2}\left(\frac{\gamma\phi}{\gamma\alpha-\alpha+\phi}\right)^2(1-2\rho_A^{n-1})\sigma_\varepsilon^2}}{\beta} - \frac{\gamma\phi\left(1-\rho_A\right)}{\gamma\alpha-\alpha+\phi} a_t \end{aligned}$$

and the $xhpr$ is

$$hpr_{n,t} - hpr_{1,t} = \left(1-\rho_A^{n-1}\right)\left(\frac{\gamma\phi}{\gamma\alpha-\alpha+\phi}\right)^2\sigma_\varepsilon^2$$

A.7 Analytical $xhpr$ and the market prices of risk

Recall that

$$\widehat{hpr}_{t,n} - \widehat{i}_t = \mathrm{Cov}_t\left[\widehat{\pi}_{t+1}, \widehat{b}_{t+1,n-1}\right] - \mathrm{Cov}_t\left[\Delta\widehat{\lambda}_{t+1}, \widehat{b}_{t+1,n-1}\right]$$

and that, up to a first order approximation, we can write $\widehat{b}_{t+1,n-1} = B_{n-1,x}'\widehat{x}_{t+1}$, $\widehat{\pi}_{t+1} = \pi_x'\widehat{x}_{t+1}$ and $\Delta\widehat{\lambda}_{t+1} = \lambda_x'\left(\widehat{x}_{t+1} - \widehat{x}_t\right)$. Since first order approximate expressions are sufficient to evaluate second moments up to a second order approximation, we obtain

$$\begin{aligned} \widehat{hpr}_{t,n} - \widehat{i}_t &= \mathrm{Cov}_t\left[\pi_x'\widehat{x}_{t+1}, B_{n-1,x}'\widehat{x}_{t+1}\right] - \mathrm{Cov}_t\left[\lambda_x'\left(\widehat{x}_{t+1} - \widehat{x}_t\right), B_{n-1,x}'\widehat{x}_{t+1}\right] \\ &= \left(\pi_x' - \lambda_x'\right)\left(\mathrm{E}_t\left[\widehat{x}_{t+1}\widehat{x}_{t+1}'\right] - \mathrm{E}_t\left[\widehat{x}_{t+1}\right]\mathrm{E}_t\left[\widehat{x}_{t+1}'\right]\right)B_{n-1,x} \\ &= \sigma^2\left(\pi_x - \lambda_x\right)'\eta\eta' B_{n-1,x} \\ &= \sigma^2 B_{n-1,x}'\eta\eta'\left(\pi_x - \lambda_x\right) \end{aligned}$$

where the last transposition can be performed because the whole expression is a scalar. Using the solution for bond prices, the expected excess holding period return can equivalently be written as

$$\begin{aligned} \widehat{hpr}_{t,n} - \widehat{i}_t &= \sigma^2 B_{1,x}'\left(I - c_1\right)^{-1}\left(I - c_1^{n-1}\right)\eta\eta'\left(\pi_x - \lambda_x\right) \\ &= \sigma^2\left(\lambda_x'\left(I - c_1\right) + \pi_x' c_1\right)\left(I - c_1\right)^{-1}\left(I - c_1^{n-1}\right)\eta\eta'\left(\pi_x - \lambda_x\right) \end{aligned}$$

This can be rewritten as

$$\widehat{hpr}_{t,n} - \widehat{i}_t = \sigma B_{n-1,x}'\eta\boldsymbol{\xi}$$

where $\boldsymbol{\xi} \equiv \sigma\eta' (\pi_x - \lambda_x)$ is an endogenous vector of "(nominal) market prices of risk". The nominal market prices of risk can be divided into market prices of real risk and of inflation risk, $\xi_\lambda \equiv -\sigma\eta'\lambda_x$ and $\xi_\pi \equiv \sigma\eta'\pi_x$, respectively.

It follows that the nominal expected excess holding period return can be written as

$$\begin{aligned}\widehat{hpr}_{t,n} - \widehat{i}_t &= \left(\widehat{hpr}^{\text{real}}_{t,n} - \widehat{r}_t\right) - \sigma\left(\lambda'_x\left(I - c_1^{n-1}\right)\eta\xi_\pi + \pi'_x c_1 (I - c_1)^{-1}\left(I - c_1^{n-1}\right)\eta\xi_\lambda\right) \\ &\quad -\sigma^2\pi'_x c_1 (I - c_1)^{-1}\left(I - c_1^{n-1}\right)\eta\eta'\pi_x\end{aligned}$$

where

$$\widehat{hpr}^{\text{real}}_{t,n} - \widehat{r}_t = -\sigma\lambda'_x\left(I - c_1^{n-1}\right)\eta\boldsymbol{\xi}_\lambda$$

and the other two components are an "excess" inflation risk premium and a convexity term. The former, more specifically, can be written as

$$\widehat{rp}_{t,n} - \widehat{rp}_{t,1} = \sigma\lambda'_x c_1^{n-1}\eta\xi_\pi - \sigma\pi'_x (I - c_1)^{-1}\left(I - c_1^{n-1}\right)c_1\eta\xi_\lambda$$

In the scalar case (setting $c_1 = \rho$ and $\eta = 1$), we obtain

$$\widehat{hpr}_{t,n} - \widehat{i}_t = (\lambda_x - \pi_x)(\lambda_x(1-\rho) + \pi_x\rho)\frac{1-\rho^{n-1}}{1-\rho}\sigma^2$$

The decomposition becomes

$$\widehat{hpr}_{t,n} - \widehat{i}_t = \lambda_x^2\left(1-\rho^{n-1}\right)\sigma^2 - \pi_x\lambda_x(1-2\rho)\frac{1-\rho^{n-1}}{1-\rho}\sigma^2 - \pi_x^2\rho\frac{1-\rho^{n-1}}{1-\rho}\sigma^2$$

This shows first that the real expected excess return is always positive, while the convexity term is always negative (for positive autocorrelation of the shocks). The sign of the nominal component depends on the sign of the term $-\lambda_x\pi_x(1-2\rho)/(1-\rho)$, namely on the sign of λ_x and π_x.

Using these general results, we can derive explicitly the market price of risk in the special cases where we obtained analytical expressions for the reduced form coefficients.

A.8 Unconditional serial correlations

Recall that second moments can be computed from the first order solution. To first order, $\mathrm{E}[\widehat{x}_t] = \mathrm{E}[\widehat{y}_t] = 0$, so that variances and covariances can be computed ignoring the squared first moment term. For example, the first order autocovariance of bond prices,

$\text{Cov}\left[\widehat{b}_{t+1,n}\widehat{b}'_{t,n}\right]$, can be computed as $\text{E}\left[\widehat{b}_{t+1,n}\widehat{b}'_{t,n}\right]$ and ignoring the term $\text{E}\left[\widehat{b}_{t+1,n}\right]\text{E}\left[\widehat{b}'_{t,n}\right]$. It follows that

$$\begin{aligned}\text{Cov}\left[\widehat{b}_{t+1,n}\widehat{b}'_{t,n}\right] &= \text{E}\left[B'_{n,x}\widehat{x}_{t+1}\widehat{x}'_t B_{n,x}\right] \\ &= B'_{n,x}\text{E}\left[\left(c_1\widehat{x}_t+\eta\sigma\varepsilon_{t+1}\right)\widehat{x}'_t\right]B_{n,x} \\ &= B'_{n,x}c_1\text{E}\left[\widehat{x}\widehat{x}'\right]B_{n,x}\end{aligned}$$

where the unconditional variance covariance matrix of $\widehat{x}_t$ is – see Hamilton (1994, p.265) – the solution of

$$\begin{aligned}\text{E}\left[\widehat{x}'\widehat{x}\right] &= \text{E}\left[\widehat{x}'c'_1c_1\widehat{x}\right]+\sigma^2\text{E}\left[\varepsilon'\eta'\eta\varepsilon\right] \\ &= \left(\text{vec}\left[\text{E}\widehat{x}\widehat{x}'\right]\right)'\text{vec}\left[c'_1c_1\right]+\sigma^2\left(\text{vec}\left[\mathbf{I}\right]\right)\text{vec}\left[\eta'\eta\right]\end{aligned}$$

namely

$$\text{vec}\left[\text{E}\left[\widehat{x}\widehat{x}'\right]\right]=\sigma^2\left(\mathbf{I}_{n_x^2}-c_1\otimes c_1\right)^{-1}\text{vec}\left[\eta'\eta\right]$$

Similarly, for the rate of growth of consumption (output)

$$\Delta\widehat{c}_{t+1}=c'_x\left(\widehat{x}_{t+1}-\widehat{x}_t\right)+\frac{1}{2}\widehat{x}'_{t+1}c_{xx}\widehat{x}_{t+1}-\frac{1}{2}\widehat{x}'_tc_{xx}\widehat{x}_t$$

we obtain

$$\begin{aligned}\text{E}\left[\Delta\widehat{c}_t\Delta\widehat{c}'_{t+1}\right] &= \text{E}\left[c'_x\left(\widehat{x}_t-\widehat{x}_{t-1}\right)\left(\widehat{x}_{t+1}-\widehat{x}_t\right)'c_x\right] \\ &= c'_x\left(2\text{E}\left[\widehat{x}\widehat{x}'_{+1}\right]-\text{E}\left[\widehat{x}\widehat{x}'_{+2}\right]-\text{E}\left[\widehat{x}\widehat{x}'\right]\right)c_x\end{aligned}$$

Hence, the first order correlation coefficient of output growth, defined as $\text{Cor}\left[\Delta\widehat{c},\Delta\widehat{c}'_{+1}\right]=\text{Cov}\left[\Delta\widehat{c},\Delta\widehat{c}'_{+1}\right]/\text{Var}\left[\widehat{c}\right]$, can be written as

$$\text{Cor}\left[\Delta\widehat{c}\Delta\widehat{c}'_{+1}\right]=-\frac{1}{2}\left(1-\frac{c'_x\left(\text{E}\left[\widehat{x}\widehat{x}'_{+1}\right]-\text{E}\left[\widehat{x}\widehat{x}'_{+2}\right]\right)c_x}{c'_x\left(\text{E}\left[\widehat{x}\widehat{x}'\right]-\text{E}\left[\widehat{x}\widehat{x}'_{+1}\right]\right)c_x}\right)$$

Noting that $\text{Cov}\left[\widehat{c}\widehat{c}'\right]\equiv\text{Var}\left[\widehat{c}\right]=\widehat{c}'_x\text{E}\left[\widehat{x}\widehat{x}'\right]\widehat{c}_x$, $\text{Cov}\left[\widehat{c}_{+1}\widehat{c}'\right]=\widehat{c}'_xc_1\text{E}\left[\widehat{x}\widehat{x}'\right]\widehat{c}_x$ and $\text{Cov}\left[\widehat{c}_{+2}\widehat{c}'\right]=\widehat{c}'_xc_1^2\text{E}\left[\widehat{x}\widehat{x}'\right]\widehat{c}_x$, it follows

$$\text{Cor}\left[\Delta\widehat{c},\Delta\widehat{c}'_{+1}\right]=-\frac{1}{2}\left(1-\frac{\text{Cov}\left[\widehat{c}',\widehat{c}_{+1}\right]-\text{Cov}\left[\widehat{c}',\widehat{c}_{+2}\right]}{\text{Var}\left[\widehat{c}\right]-\text{Cov}\left[\widehat{c}',\widehat{c}_{+1}\right]}\right)$$

Note that the autocovariance of output is a scalar. We can then divide through by its variance to rewrite the above expression in terms of the correlation coefficients of the *level* of output

$$\text{Cor}\left[\Delta\widehat{c},\Delta\widehat{c}'_{+1}\right]=-\frac{1}{2}\left(1-\frac{\text{Cor}\left[\widehat{c}',\widehat{c}_{+1}\right]-\text{Cor}\left[\widehat{c}',\widehat{c}_{+2}\right]}{1-\text{Cor}\left[\widehat{c}',\widehat{c}_{+1}\right]}\right)$$

If we define $\rho_j \equiv \text{Cor}\left[\widehat{c}_{+j}, \widehat{c}'\right]$, we can alternatively write

$$\text{Cor}\left[\Delta\widehat{c}, \Delta\widehat{c}'_{+1}\right] = -\frac{1}{2}\left(1 - \frac{\rho_1 - \rho_2}{1 - \rho_1}\right)$$

which is positive when $\rho_1 - \rho_2 > 1 - \rho_1$, or $\rho_2 < 2\rho_1 - 1$. In other words, the drop in the first autocorrelation coefficient of output must be smaller than that from the first to the second.

To develop an intuition for this condition, it is useful to translate it in terms of the parameters of the equilibrium process followed by consumption. In practice, the parameters of the process will be determined endogenously, but it is useful to consider a few special cases focusing on the general form of the driving process.

A.8.1 Consumption follows an AR(1) process

Consumption follows a stationary AR(1) process in the simple model without habit persistence and with flexible prices analysed in the text. Under this assumption, the condition for a positive serial correlation of consumption growth is never satisfied. If the process is written as $\widehat{c}_t = \phi\widehat{c}_{t-1} + \varepsilon_t$, the correlation coefficients of the consumption level would in fact be $\text{Cor}\left[\widehat{c}_{+1}\widehat{c}'\right] = \phi$, $\text{Cor}\left[\widehat{c}_{+2}\widehat{c}'\right] = \phi^2$. It follows that

$$\text{Cor}\left[\Delta\widehat{c}, \Delta\widehat{c}'_{+1}\right] = -\frac{1}{2}(1 - \phi) > 0$$

is never satisfied.

A.8.2 Consumption follows an AR(2) process

This is just a generalisation of the former case, where $\widehat{c}_t = \phi_1\widehat{c}_{t-1} + \phi_2\widehat{c}_{t-1} + \varepsilon_t$. The first order autocorrelations are

$$\begin{aligned} \rho_1 &= \frac{\phi_1}{1 - \phi_2} \\ \rho_2 &= \frac{\phi_1^2}{1 - \phi_2} + \phi_2 \end{aligned}$$

so that $\text{Cor}\left[\Delta\widehat{c}, \Delta\widehat{c}'_{+1}\right] > 0$ implies

$$\frac{\phi_1^2}{1 - \phi_2} + \frac{\phi_2 - \phi_2^2}{1 - \phi_2} < 2\frac{\phi_1}{1 - \phi_2} - \frac{1 - \phi_2}{1 - \phi_2}$$

If $\phi_2 < 1$, this implies

$$\phi_1 + \phi_2 > 1$$

while if $\phi_2 > 1$, it must be the case that

$$\phi_1 + \phi_2 < 1$$

The process is stationary if the roots $\frac{\phi_1 \pm \sqrt{\phi_1^2 + 4\phi_2}}{2} < 1$, which implies $\phi_1 + \phi_2 < 1$ and $\phi_2 - \phi_1 < 1$ (see Hamilton, 1994). This excludes the case $\phi_2 < 1$. When $\phi_2 > 1$, it must be the case that $\phi_1 < 1 - \phi_2$. Combined with the first stationarity requirement, this implies $\phi_2 - 1 < \phi_1 < 1 - \phi_2$, or $\phi_2 < 1$, which is impossible given the assumption $\phi_2 > 1$.

A.8.3 Consumption follows an MA(1) process

If consumption follows the process $\widehat{c}_t = \varepsilon_t + \theta\varepsilon_{t-1}$, the autororrelations are

$$\begin{aligned} \rho_1 &= \frac{\theta}{1+\theta^2} \\ \rho_2 &= 0 \end{aligned}$$

The condition $\text{Cor}\left[\Delta\widehat{c}, \Delta\widehat{c}'_{+1}\right] > 0$ implies

$$\frac{1}{2}\theta^2 - \theta + \frac{1}{2} < 0$$

which is satisfied for values of θ between the solutions of the equation $\frac{1}{2}\theta^2 - \theta + \frac{1}{2} = 0$. This equation, however, has coincident solutions $\theta = 1$. The condition is therefore never satisfied

A.8.4 Consumption follows an MA(2) process

If $\widehat{c}_t = \varepsilon_t + \theta_1\varepsilon_{t-1} + \theta_2\varepsilon_{t-2}$, the autocorrelations are

$$\begin{aligned} \rho_1 &= \frac{\theta_1(1+\theta_2)}{1+\theta_1^2+\theta_2^2} \\ \rho_2 &= \frac{\theta_2}{1+\theta_1^2+\theta_2^2} \end{aligned}$$

The condition $\text{Cor}\left[\Delta\widehat{c}, \Delta\widehat{c}'_{+1}\right] > 0$ implies

$$\frac{\theta_2}{1+\theta_1^2+\theta_2^2} < 2\frac{\theta_1(1+\theta_2)}{1+\theta_1^2+\theta_2^2} - 1$$

or, since $1+\theta_1^2+\theta_2^2 > 0$,

$$\theta_1^2 - 2\theta_1(1+\theta_2) + \theta_2(1+\theta_2) + 1 < 0$$

The solution of $\theta_1^2 - 2\theta_1(1+\theta_2) + \theta_2(1+\theta_2) + 1 = 0$ are $\theta_1 = 1 + \theta_2 \pm \sqrt{\theta_2}$. Hence, the above inequality is satisfied when

$$1 + \theta_2 - \sqrt{\theta_2} < \theta_1 < 1 + \theta_2 + \sqrt{\theta_2}$$

A.8.5 Consumption follows an ARMA(1,1) process

The ARMA case is the most interesting one, because consumption will in general follow such a process in DSGE models. For simplicity, here we focus on the ARMA(1,1) case with AR parameter ϕ and MA parameter θ.

The first and second order autocorrelations are given by

$$\rho_1 = \frac{(\phi+\theta)(1+\theta\phi)}{1+2\theta\phi+\theta^2}$$
$$\rho_2 = \phi\frac{(\phi+\theta)(1+\theta\phi)}{1+2\theta\phi+\theta^2}$$

Hence, the condition Cor $\left[\Delta\widehat{c}, \Delta\widehat{c}'_{+1}\right] > 0$ implies

$$\frac{(2-\phi)(\phi+\theta)(1+\theta\phi)}{1+2\theta\phi+\theta^2} > 1$$

The denominator $1+2\theta\phi+\theta^2 > 0$ for a stationary process with $\phi < 1$, because the solution of $1+2\theta\phi+\theta^2 = 0$ is complex. It follows that

$$(2-\phi)(\phi+\theta)(1+\theta\phi) > 1+2\theta\phi+\theta^2$$

or

$$(1-\phi)^2+\theta^2(1-\phi)^2-(1-\phi)(2-\phi(1-\phi))\theta < 0$$

or

$$\theta^2(1-\phi)-\theta(2-\phi(1-\phi))+(1-\phi) < 0$$

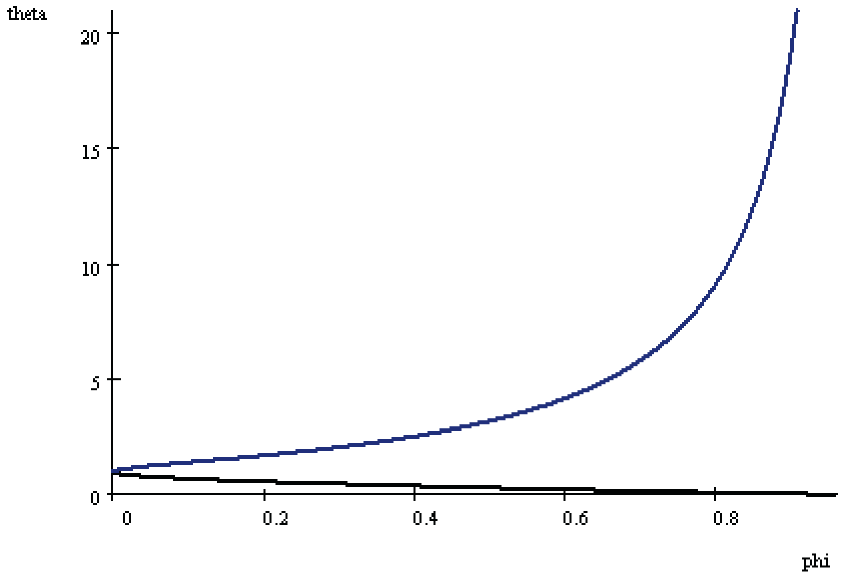
This inequality is satisfied when

$$\frac{2-\phi(1-\phi)-\sqrt{\phi\left(4+\phi(1-\phi)^2\right)}}{2(1-\phi)} < \theta < \frac{2-\phi(1-\phi)+\sqrt{\phi\left(4+\phi(1-\phi)^2\right)}}{2(1-\phi)}$$

These inequalities are never satisfied when $\phi\left(4+\phi(1-\phi)^2\right) < 0$, i.e. when $\phi < 0$. The lower bound is positive if $(1-\phi)^2 > 0$, which is always satisfied. Hence θ must be positive. More specifically, it must be comprised within the two lines in Figure 5.

The sum of the lower bound for θ plus ϕ, i.e. $\frac{2-\phi(1-\phi)-\sqrt{\phi\left(4+\phi(1-\phi)^2\right)}}{2(1-\phi)}+\phi$, is always greater than 1, given that this requires $\phi > 0$. Hence, a sufficient condition for consumption growth to be positively serially correlated is $\theta+\phi > 1$, which also implies a hump-shaped response to shocks.

Figure 5: Combination of the parameters ϕ and θ which ensure a positive serial correlation of consumption growth.



References

[1] Ascari, G. (2004), "Staggered Prices and Trend Inflation: Some Nuisances," Review of Economic Dynamics 7, 642-667.

[2] Backus, D. K., A. W. Gregory and S. E. Zin (1989), "Risk premiums in the term structure," Journal of Monetary Economics 24, 371-99.

[3] Bekaert, G., S. Cho, and A. Moreno (2006), "New-Keynesian Macroeconomics and the Term Structure", mimeo, Columbia University.

[4] Benigno, P. and M. Woodford (2005), "Inflation stabilisation and welfare: the case of a distorted steady state", Journal of the European Economic Association 3, 1185:1236.

[5] Boudoukh, J. (1993), "An equilibrium model of nominal bond prices with inflation-output correlation and stochastic volatility," Journal of Money, Credit, and Banking 25, 636-665.

[6] Buraschi, A. and A. Jiltsov (2005), "Inflation risk premia and the expectations hypothesis," Journal of Financial Economics 75, 429–490.

[7] Buraschi, A. and A. Jiltsov (2007), "Term Structure of Interest Rates Implications of Habit Persistence," The Journal of Finance, forthcoming.

[8] Calvo, G. A. (1983), "Staggered prices in a utility-maximising framework," Journal of Monetary Economics 12, 983-98.

[9] Campbell, J. Y. and J. H. Cochrane (1999), "By Force of Habit: A Consumption-Based Explanation of Aggregate Stock Market Behavior", Journal of Political Economy 107:205-251.

[10] Canova, F. and J. Marrinan (1996), "Reconciling the term structure of interest rates with the consumption-based ICAP model", Journal of Economic Dynamics and Control 20, 709-750.

[11] Chapman, D. A. (1997), "The cyclical properties of consumption growth and the real term structure", Journal of Monetary Economics 39:145-172.

[12] Christiano, L., M. Eichenbaum and C. L. Evans (2005), "Nominal Rigidities and the Dynamic Effects of a Shock to Monetary Policy," Journal of Political Economy 113, 1-45.

[13] Clarida, R., J. Galí and M. Gertler (2000), "Monetary Policy Rules and Macroeconomic Stability: Evidence and Some Theory," Quarterly Journal of Economics 147-180, February.

[14] Constantinides, G. M. (1990), "Habit formation: a resolution of the equity premium puzzle," Journal of Political Economy 98, 519-43.

[15] Cox, J., J. Ingersoll and S. Ross (1985), "A theory of the term structure of interest rates," Econometrica 53, 385–408.

[16] Dai, Q. (2003), "Term structure dynamics in a model with stochastic internal habit," mimeo.

[17] Dai, Q. and K.J. Singleton (2000), "Specification analysis of affine term structure models," Journal of Finance, 55, 1943-1978.

[18] Dai, Q. and K. Singleton (2002), "Expectation Puzzles, Time-varying Risk Premia, and Affine Models of the Term Structure," Journal of Financial Economics 63, 415-441.

[19] Den Haan, W. J. (1995), "The term structure of interest rates in real and monetary economies," Journal of Economic Dynamics and Control 19, 909-40.

[20] Donaldson, J. B., T. Johnsen and R. Mehra (1990), "On the term structure of interest rates," Journal of Economic Dynamics and Control 14, 571-96.

[21] Duffee, G.R. (2002), "Term Premia and Interest Rate Forecasts in Affine Models," Journal of Finance 57, 405-43.

[22] Duffie, D. and R. Kan (1996), "A yield-factor model of interest rates," Mathematical Finance 6, 379-406.

[23] Fuhrer, J. (2000), "Habit Formation in Consumption and Its Implications for Monetary-Policy Models," American Economic Review 90, 367-390.

[24] Gallmeyer, M.F., B. Hollifield and S.E. Zin (2005), "Taylor rules, McCallum rules and the term structure of interest rates," *Journal of Monetary Economics* 52, 921-950.

[25] Hamilton, J. (1994), Time series analysis, (Princeton University Press, Princeton).

[26] Hördahl, P., O. Tristani and D. Vestin (2006), "A joint econometric model of macroeconomic and term structure dynamics," *Journal of Econometrics*, 131, 405-444.

[27] Hördahl, P., O. Tristani and D. Vestin (2007), "The yield curve and macroeconomic dynamics," ECB Working Paper.

[28] Judd, K. (1998), Numerical Methods in Economics, Cambridge (MA): MIT Press.

[29] Mehra, R. and E.C. Prescott (1985), "The equity premium: A puzzle," Journal of Monetary Economics 15, 145-61.

[30] Orphanides, A. (2004), "Monetary policy rules, macroeconomic stability and inflation: a view from the trenches," Journal of Money, Credit and Banking 36, 151-175.

[31] Orphanides, A. and J.C. Williams (2002), "Robust Monetary Policy Rules with Unknown Natural Rates", Brookings Papers on Economic Activity 2, 63-118.

[32] Piazzesi, M. and M. Schneider (2006), "Equilibrium Yield Curves," in D. Acemoglu, K. Rogoff, and M. Woodford, *NBER Macroeconomics Annual 2006*, Cambridge MA: MIT press, p. 389-442.

[33] Ravenna, F. and J. Seppälä (2005), "Monetary Policy and the Term Structure of Interest Rates," mimeo, University of Illinois, February.

[34] Rudebusch, G. and T. Wu (2007), "A Macro-Finance Model of the Term Structure, Monetary Policy, and the Economy," Economic Journal, forthcoming.

[35] Schmitt-Grohé and Uribe (2004), "Solving dynamic general equilibrium models using a second-order approximation to the policy function," Journal of Economic Dynamics and Control 28, 755-75.

[36] Smets, F. and R. Wouters (2003), "An estimated stochastic dynamic general equilibrium model of the euro area," Journal of European Economic Association 1, 1123-75

[37] Smets, F. and R. Wouters (2005), "Comparing shocks and frictions in US and euro area business cycles: a Bayesian DSGE approach," Journal of Applied Econometrics 20, 161-183.

[38] Sutherland, A. (2002), "A simple second-order solution method for dynamic general equilibrium models," CEPR Discussion Paper No. 3554.

[39] Taylor, J. B. (1993), "Discretion vs. policy rules in practice," Carnegie-Rochester Conference Series on Public Policy 39, 195-214.

[40] Wachter, J. (2006), "A Consumption-Based Model of the Term Structure of Interest Rates," Journal of Financial Economics 79, 365-399.

[41] Woodford, M. (2003), Interest and Prices, Princeton University Press.

[42] Wu, T. (2006), "Macro Factors and the Affine Term Structure of Interest Rates," Journal of Money, Credit, and Banking 38, 1847-1875.

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