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Forecasting and stress testing with quantile vector autoregression

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Abstract

We introduce a structural quantile vector autoregressive (VAR) model. Unlike standard VAR which models only the average interaction of the endogenous variables, quantile VAR models their interaction at any quantile. We show how to estimate and forecast multivariate quantiles within a recursive structural system. The model is estimated using real and financial variables. The dynamic properties of the system change across quantiles. This is relevant for stress testing exercises, whose goal is to forecast the tail behavior of the economy when hit by large financial and real shocks.

Keywords: Regression quantiles; Structural VAR; Growth at Risk.

JEL Codes: C32; C53; E17; E32; E44.

NON-TECHNICAL SUMMARY

The standard definition of financial stability adopted by central banks around the world emphasises the negative impact that severe financial shocks may have on real economic activity. This definition underscores an intrinsic tension in connecting the macro and financial dimensions of the economy. The empirical workhorse of macroeconomists is the vector autoregressive (VAR) model, which studies the expected dynamics of the endogenous variables. Financial instability, on the other hand, is inherently linked to the tail dynamics of the system. Using econometric models developed to analyse the average behaviour of macroeconomic variables is bound to miss important aspects of macro-financial linkages which arguably only arise when the system is affected by tail shocks. This paper develops a quantile VAR model, which is designed to address many of the core questions of the macro-finance research agenda.

Quantile VAR models the interaction and feedback effects that the variables of the system have on their quantile dynamics. To study the macro-financial linkages in Europe, we estimate a quantile VAR model on euro area data for industrial production and an indicator of financial distress. We find that financial shocks – defined as a tail quantile realization – are transmitted to the real economy only when the economy is simultaneously hit by a real negative shock. Modelling the mean dynamics with a standard VAR misses most of the action associated with this important channel of transmission of financial shocks. Furthermore, shutting down financial linkages in the system significantly changes the dynamics of the real economy when hit by negative shocks, but leaves the dynamics largely unaffected in normal conditions. One advantage of quantile VAR is that it allows us to perform impulse response analyses and to forecast the quantiles of the endogenous variables. We find that by hitting the system with a financial shock there is a strong and persistent asymmetric impact on the distribution of industrial production, which takes about two years to be absorbed.

Quantile VAR provides also the natural environment to perform stress testing exercises. To its core, stress testing is a forecast of what happens to the system when it is hit by an arbitrary sequence of negative shocks. If the euro area is hit by a sequence of six monthly consecutive financial and real shocks, its industrial production contracts by a cumulated amount of more than 10% over the same period. This contrasts with a median increase of industrial production of around 2%, a forecast which would hold under normal circumstances.

This paper also contributes to the quantile regression econometric literature by showing how to deal with multiple variables and how to forecast in a time series context. Our econometric framework is general enough to cover the modelling of multiple quantiles of multiple random variables. Stress testing can be thought of as an estimate of the reaction of the endogenous random variables when the system is hit by a sequence of quantile shocks. Stress scenarios are nothing else than an arbitrary series of quantile shocks hitting the macro-financial environment.

1 Introduction

Vector autoregressive (VAR) models are the empirical workhorse of macroeconomists. In their most basic formulation, these models rely on constant coefficients and i.i.d. Gaussian innovations. There is, however, substantial empirical evidence that macroeconomic variables are characterised by nonlinearities and asymmetries which cannot be captured by simple linear Gaussian models (Perez-Quiros and Timmermann 2000, Hubrich and Tetlow 2015, Kilian and Vigfusson 2017, Adrian, Boyarchenko and Giannone 2019). We show how to estimate recursive structural VAR models with quantile regression methods. The insights of our approach can be extended more generally to produce iterated forecasts for nonlinear models. The methodology is applied to the euro area, revealing the presence of an asymmetric and sizable downside risk to the real economy, driven by shocks to the financial system.

Quantile regression was introduced by Koenker and Bassett (1978) and has found many applications in economics (Koenker 2005, 2017). Early applications to univariate time series include Engle and Manganelli (2004) and Koenker and Xiao (2006). White, Kim and Manganelli (2010, 2015) develop a framework to model multivariate quantiles. Schüler (2014) introduces a Bayesian quantile structural vector autoregressive model. In homoskedastic linear regression models, the conditioning variables shift the location of the conditional density of the dependent variables, but they have no effect on conditional dispersion or shape. In general, however, this need not be the case. Quantile regression is a semiparametric technique which allows different covariates to affect different parts of the distribution. If and how this happens is an empirical question. In our empirical applications, we find that estimates of quantile regression slopes and quantile impulse response functions vary across quantiles. This may happen either because of unmodeled time varying higher order moments, and/or because the conditioning variables affect the conditional distribution of the dependent variables in a nonlinear way. These effects cannot be detected with standard OLS VAR estimates.

The VAR for VaR model of White et al. (2015) represents the starting point of our analysis, as it provides the general framework for inference. Casting the problem in a multivariate framework such as a VAR model immediately raises the issue of the definition of structural shocks and identification. We show that structural identification and quantile modeling of multiple variables are different sides of the same coin. We identify the quantile VAR by estimating a recursive system, where the variables ordered earlier are allowed to contemporaneously affect the remaining variables, as in the recursive conditioning framework of Chesher (2003).

This recursion implies that the quantile of the second variable conditional on the first one, say, is a random variable as it depends on the contemporaneous value taken by the first variable. By conditioning on the quantile of the first random variable (which is available from the previous equation of the system) we can estimate the quantile of the second random variable. This reasoning can be repeated recursively for all the random variables, therefore giving the quantile of the quantile of the cross section at any given point in time.

This intuition holds also for multi step ahead quantile forecasting. The quantile two periods ahead depends on the value taken by the random variables one period ahead. By conditioning on the quantile values of the random variables one period ahead (which is available from the initial one step ahead forecast described in the previous paragraph), we can estimate the quantiles of the two step ahead random variables. Iterating this reasoning forward, we can obtain any multi step ahead quantile forecast.

Our econometric framework is general enough to cover the modeling of multiple quantiles of multiple random variables. It is this multivariate approach that gives the flexibility to assess the impact of any future quantile realization. Stress testing can be thought of as an estimate of the reaction of the endogenous random variables when the system is hit by a sequence of tail shocks, where tail shocks are defined by future realizations of the random variables being equal to low quantiles. Stress scenarios are defined as an arbitrary series (to be chosen by the policy maker or calibrated to past crises) of future quantile realizations hitting the system.

We estimate a quantile VAR model on euro area data for industrial production growth and an indicator of financial distress and perform three types of exercises.

First, we estimate euro area growth at risk, defined as the 10% quantile of industrial production growth. We find that severe financial shocks are transmitted to the real economy only when the economy is simultaneously hit by a real negative shock. Modeling the conditional mean with a standard VAR misses most of these dynamics. Furthermore, shutting down the financial channel of transmission in the system significantly changes the dynamics of the real economy when hit by negative shocks, but leaves the dynamics largely unaffected in normal conditions. These results are broadly in line with those found by Adrian et al. (2019) for the U.S. economy. The empirical model estimated by Adrian et al. (2019) is equivalent to estimating only one equation of our quantile VAR model. The advantage of quantile VAR is that it allows us to perform impulse response analyses and to forecast the quantiles of the endogenous variables. We find that by hitting the system with a financial shock there is a strong, persistent and asymmetric impact on the distribution of industrial production, which takes about two years to be absorbed.

Second, we forecast euro area growth under alternative stress scenarios. Quantile VAR provides the natural environment to perform stress testing exercises. At its core, stress testing is a forecast of what happens to the system when it is hit by an arbitrary sequence of negative shocks. If the euro area is hit by a sequence of six monthly consecutive financial and real 10% quantile realizations, its industrial production contracts by a cumulated amount of more than 10% over the same period. This contrasts with a median forecast (that is, a sequence of median realizations of the endogenous variables) of industrial production of around 2%.

Third, we perform a counterfactual scenario analysis before Lehman Brothers' default. Using estimates up to August 2008, we find evidence of sizable and unprecedented downside risk to the euro area real economy already in mid 2007. Such counterfactual exercises require the estimation of a structural quantile model.

The paper is organized as follows. Section 2 develops the general quantile structural vector autoregressive framework. It provides the links with standard OLS structural VAR, derives the asymptotic distributions, and shows how to do forecasting with quantile structural VAR. Section 3 estimates the quantile VAR model for the euro area, and performs a stress testing exercise and the counterfactual analysis before Lehman's bankruptcy. Section 4 concludes.

2 Quantile Vector Autoregression

This section introduces the structural quantile VAR (QVAR) and impulse response function, shows how to compute quantile VAR forecasts and provides the asymptotic properties of the model.

2.1 The Law of Iterated Quantiles and Quantile Impulse Response Functions

Consider a sequence of random variables $\{\tilde{Y}_t : t = 1, \ldots, T\}$, where $\tilde{Y}_t$ is an $n \times 1$ vector with i^{th} element denoted by $\tilde{Y}_{it}$ for $i \in \{1, \ldots, n\}$.

Consider the following structural vector autoregressive model:

$$\tilde{Y}_{t+1} = \omega + A_0\tilde{Y}_{t+1} + A_1\tilde{Y}_t + \epsilon_{t+1}, \qquad \epsilon_{t+1} \sim i.i.d.(0, \Sigma) \tag{1}$$

where A_0 and A_1 are a $n \times n$ coefficient matrices, ω is a $n \times 1$ vector of constants, ϵ_{t+1} is a $n \times 1$ vector of i.i.d. structural zero mean shocks with Σ a diagonal matrix. Imposing that A_0 has a lower triangular structure with zeros along the main diagonal, the identification of this system is equivalent to assuming a Choleski decomposition of the variance covariance matrix of the residuals from a standard reduced form VAR (see, for instance, chapter 2 of Lütkepohl 2005).

Denote with $E_t(\tilde{Y}_{t+H+1}) \equiv E(\tilde{Y}_{t+H+1}|\Omega_t)$ the expected value of $\tilde{Y}_{t+H+1}$ given Ω_t, the information available at time t. We report for convenience the following result.

Theorem 1 ***(VAR forecast)*** — *The mean forecast at time t of process (1) for $H \geq 0$ is:*

$$E_t(\tilde{Y}_{t+H+1}) = \sum_{h=0}^{H} B^h\nu + B^{H+1}\tilde{Y}_t \tag{2}$$

where $\nu \equiv (I_n - A_0)^{-1}\omega$, $B \equiv (I_n - A_0)^{-1}A_1$ and I_n is the identity matrix of *dimension n.*

Proof — See appendix.

The impulse-response function is derived from the marginal impact that a structural shock has on the expected value of future expectations, via the impact it has on $\tilde{Y}_t$.

Theorem 2 ***(Mean Impulse Response Function)*** — *The response of the mean forecast of process* (1) *to a unit structural shock to the variables of the system is:*

$$\partial E_t(\tilde{Y}_{t+H+1})/\partial \epsilon_t' = B^{H+1}(I_n - A_0)^{-1}, \quad \text{for } H \geq 0 \tag{3}$$

Proof — See appendix.

This framework motivates our definition of a structural QVAR. Since we want to consider the possibility of jointly modelling multiple quantiles, we need additional notation. For our purposes, it is important to define a recursive information set, which allows us to work with structural models.

Definition 1 ***(Recursive information set)*** — *The recursive information set at time t is defined as:*

$\Omega_{1t} \equiv \{\tilde{Y}_t, \tilde{Y}_{t-1}, \ldots\}$

$\Omega_{it} \equiv \{\tilde{Y}_{1,t+1}, \ldots, \tilde{Y}_{i-1,t+1}, \tilde{Y}_t, \tilde{Y}_{t-1}, \ldots\}$ *for* $i = 2, \ldots, n$.

According to this definition, the recursive information set Ω_{2t}, say, contains all the lagged values of $\tilde{Y}_t$ as well as the contemporaneous value of $\tilde{Y}_{1,t+1}$.

Considering p distinct quantiles, $0 < \theta_1 < \theta_2 < \ldots < \theta_p < 1$, the quantile structural vector autoregressive model is defined as follows:[1]

$$Y_{t+1} = \omega^\theta + A_0^\theta Y_{t+1} + A_1^\theta Y_t + \epsilon_{t+1}^\theta, \quad P(\epsilon_{i,t+1}^{\theta_j} < 0 | \Omega_{it}) = \theta_j, \tag{4}$$
$$i = 1, \ldots, n, \quad j = 1, \ldots, p$$

The dependent variable Y_t is now an np-vector, which is obtained as $Y_t = \iota_p \otimes \tilde{Y}_t$, where ι_p is a p-vector of ones, and $\epsilon_t^\theta \equiv [\epsilon_{1t}^{\theta_1}, \ldots, \epsilon_{nt}^{\theta_1}, \ldots, \epsilon_{1t}^{\theta_p}, \ldots, \epsilon_{nt}^{\theta_p}]'$. The matrices A_0^θ and A_1^θ are block diagonal, to avoid trivial multicollinearity problems. We further impose that the diagonal blocks of A_0^θ are lower triangular matrices with zeros along their main diagonal, reflecting the recursive identification assumption of the system. The conditional quantile restriction on the residuals defining the regression quantile follows the recursive structure of the identification assumption.

To derive the results for the quantile VAR forecast, we need first to define the quantile operators.

Definition 2 ***(Quantile operators)*** — *Given the an $np \times 1$ random vector X_t and a nonrandom $np \times np$ matrix A, let $Z_t \equiv AX_t$, with individual elements denoted by Z_t^i, $i = 1, \ldots, np$.*

[1] The model can be generalised to the case where quantile indices are different for different elements of Y_t. See White et al. (2015) for details.

The ***cross sectional conditional quantile*** *is defined by:*

$$Q_t^\theta(Z_{t+1}) \equiv [Q_{1t}^{\theta_1}(\dots Q_{nt}^{\theta_1}(Z_{t+1}^1)), Q_{1t}^{\theta_1}(\dots Q_{nt}^{\theta_1}(Z_{t+1}^2)), \dots, Q_{1t}^{\theta_1}(\dots Q_{nt}^{\theta_1}(Z_{t+1}^n)),$$
$$\dots,$$
$$Q_{1t}^{\theta_p}(\dots Q_{nt}^{\theta_p}(Z_{t+1}^{n(p-1)+1})), Q_{1t}^{\theta_p}(\dots Q_{nt}^{\theta_p}(Z_{t+1}^{n(p-1)+2})), \dots, Q_{1t}^{\theta_p}(\dots Q_{nt}^{\theta_p}(Z_{t+1}^{np}))]'$$

where the generic ***conditional quantile*** $Q_{kt}^{\theta_j}(Z_{t+1}^i)$*, given the information set* Ω_{kt}*, is implicitly defined by:*

$$P(Z_{t+1}^i < Q_{kt}^{\theta_j}(Z_{t+1}^i)|\Omega_{kt}) = \theta_j \quad \textit{for } j = 1, \dots, p \quad k = 1, \dots, n$$

The conditional quantile corresponds to the standard definition of quantile, conditional on the appropriate information set. The cross sectional conditional quantile is an operator that for each element of a vector takes the conditional quantiles of the conditional quantile for all the cross sectional variables, using the recursive information set.

The following theorems derive the quantile forecast and the quantile impulse response function.

Theorem 3 ***(Quantile VAR forecast)*** — *The quantile forecast at time* t *of process* (4) *for* $H \geq 0$*, is:*

$$Q_t^\theta(\cdots Q_{t+H}^\theta(Y_{t+H+1})) = \sum_{h=0}^{H}(B^\theta)^h\nu^\theta + (B^\theta)^{H+1}Y_t \tag{5}$$

where $\nu^\theta \equiv (I_{np} - A_0^\theta)^{-1}\omega^\theta$ *and* $B^\theta \equiv (I_{np} - A_0^\theta)^{-1}A_1^\theta$.

Proof — See appendix.

The quantile impulse response function is derived from the marginal impact that a structural shock has on the quantile of future quantiles, via the impact it has on $\tilde{Y}_t$.

Theorem 4 ***(Quantile Impulse Response Function)*** — *The response of the quantile forecast of process* (1) *to a unit structural shock to the variables of the system is:*

$$\partial Q_t^\theta(\cdots Q_{t+H}^\theta(Y_{t+H+1}^\theta))/\partial(\epsilon_t^\theta)' = (B^\theta)^{H+1}(I_{np} - A_0^\theta)^{-1} \quad \text{for } H \geq 0 \tag{6}$$

Proof — See appendix.

An explicit example may help. Consider a model with two endogenous random variables and two quantiles, say 50% and 90%. System (4) can be written explicitly as:

$$\begin{bmatrix} \tilde{Y}_{1,t+1} \\ \tilde{Y}_{2,t+1} \\ \hdashline \tilde{Y}_{1,t+1} \\ \tilde{Y}_{2,t+1} \end{bmatrix} = \begin{bmatrix} \omega_1^{.5} \\ \omega_2^{.5} \\ \hdashline \omega_1^{.9} \\ \omega_2^{.9} \end{bmatrix} + \left[\begin{array}{cc:cc} 0 & 0 & 0 & 0 \\ a_{021}^{.5} & 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 & 0 \\ 0 & 0 & a_{021}^{.9} & 0 \end{array}\right] \begin{bmatrix} \tilde{Y}_{1,t+1} \\ \tilde{Y}_{2,t+1} \\ \hdashline \tilde{Y}_{1,t+1} \\ \tilde{Y}_{2,t+1} \end{bmatrix} + \tag{7}$$
$$+ \left[\begin{array}{cc:cc} a_{11}^{.5} & a_{12}^{.5} & 0 & 0 \\ a_{21}^{.5} & a_{22}^{.5} & 0 & 0 \\ \hdashline 0 & 0 & a_{11}^{.9} & a_{12}^{.9} \\ 0 & 0 & a_{21}^{.9} & a_{22}^{.9} \end{array}\right] \begin{bmatrix} \tilde{Y}_{1t} \\ \tilde{Y}_{2t} \\ \hdashline \tilde{Y}_{1t} \\ \tilde{Y}_{2t} \end{bmatrix} + \begin{bmatrix} \epsilon_{1,t+1}^{.5} \\ \epsilon_{2,t+1}^{.5} \\ \hdashline \epsilon_{1,t+1}^{.9} \\ \epsilon_{2,t+1}^{.9} \end{bmatrix}$$

That is, model (4) is a convenient notation to stack all the estimated QVAR models. Let us work with this example to illustrate the intuition of the proof of theorem 3. The line corresponding to the 90%, say, quantile of $\tilde{Y}_{2,t+1}$ is:

$$\tilde{Y}_{2,t+1} = q_{2t}^{.9} + a_{021}^{.9}\epsilon_{1,t+1}^{.9} + \epsilon_{2,t+1}^{.9}$$

where $q_{2t}^{.9} \equiv \omega_2^{.9} + a_{021}^{.9}\omega_1^{.9} + (a_{021}^{.9}a_{11}^{.9} + a_{21}^{.9})\tilde{Y}_{1t} + (a_{021}^{.9}a_{12}^{.9} + a_{22}^{.9})\tilde{Y}_{2t}$. The 90% quantile of $\tilde{Y}_{2,t+1}$ conditional on the information set $\Omega_{2,t} = \{\tilde{Y}_{1,t+1}, \tilde{Y}_t\}$ is $Q_{2t}^{.9}(\tilde{Y}_{2,t+1}) = q_{2t}^{.9} + a_{021}^{.9}\epsilon_{1,t+1}^{.9}$, because, by the conditional quantile restriction of (4), $Q_{2t}^{.9}(\epsilon_{2,t+1}^{.9}) = 0$. This quantity is still a random variable at time t, because of the term $a_{021}^{.9}\epsilon_{1,t+1}^{.9}$. The 90% quantile of $Q_{2t}^{.9}(\tilde{Y}_{2,t+1})$, conditional on the information set $\Omega_{1,t} = \{\tilde{Y}_t\}$ is therefore $Q_{1t}^{.9}(Q_{2t}^{.9}(\tilde{Y}_{2,t+1})) = q_{2t}^{.9}$. This reasoning can be repeated for any n variables in Y_{t+1} and for any future variables Y_{t+h} for any $h \geq 1$.

If system (1) is the data generating process, then $\omega^\theta = \iota_p \otimes \omega + \kappa^\theta$, where κ^θ is the np-vector containing the θ quantiles of ϵ_{t+1}, $A_0^\theta = I_p \otimes A_0$ and $A_1^\theta = I_p \otimes A_1$. Under this assumption, the VAR and quantile VAR are characterized by identical dynamics.

We refer to equation (5) as to the *Law of Iterated Quantiles*. Notice the difference with respect to the law of iterated expectations, which states that, given any generic random variable X_t with finite expectation:

$$E_t(X_{t+1} + X_{t+2}) = E_t(E_{t+1}(X_{t+1} + X_{t+2}))$$

For the law of iterated quantiles, instead, this is generally not the case:

$$Q_t^\theta(X_{t+1} + X_{t+2}) \neq Q_t^\theta(Q_{t+1}^\theta(X_{t+1} + X_{t+2}))$$

2.2 Forecasting and stress testing

The result of theorem 3 is restricted to the quantiles of the same quantile. That is, it derives the forecast for the 90% quantile, say, of future and cross sectional 90% quantiles. This subsection generalizes this result, by deriving the quantile forecast for any combination of future quantiles.

Define $S_{j_{t+1}}$ the $n \times np$ matrix selecting specific quantiles from the vector ϵ_{t+1}^θ. That is, $S_{j_{t+1}}\epsilon_{t+1}^\theta = [\epsilon_{1,t+1}^{\theta_{j_{t+1}^1}}, \ldots, \epsilon_{n,t+1}^{\theta_{j_{t+1}^n}}]'$ for $j_{t+1}^i \in \{1, \ldots, p\}$ and $i \in \{1, \ldots, n\}$. The superscripts of the structural residuals are indexed by j_{t+h}^i, as this gives the flexibility to choose any quantile (indexed by j) of any of the endogenous variables (indexed by i) at any future period (indexed by $t+h$).

Theorem 5 ***(Generic quantile VAR forecast)*** — *Let $\{S_{j_{t+h}}\}_{h=1}^{H}$ denote the sequence of matrices selecting the future quantiles to be forecasted and $\hat{Y}_{t+H}$ the value of the dependent variable corresponding to this sequence of quantile realization. The corresponding quantile forecasts as of time t of process (4) for $H \geq 1$, can be computed recursively as:*

$$\hat{Y}_{t+1} = \bar{B}^{\theta}_{t+1}(\omega^{\theta} + A^{\theta}_{1} Y_t) \tag{8}$$

$$\hat{Y}_{t+H} = \bar{B}^{\theta}_{t+H}(\omega^{\theta} + A^{\theta}_{1} \hat{Y}_{t+H-1}) \qquad \text{for } H \geq 2 \tag{9}$$

where $\bar{B}^{\theta}_{t+H} \equiv (I_n - S_{j_{t+H}} A^{\theta}_{0} \bar{S})^{-1} S_{j_{t+H}}$ and $\bar{S}$ is the $pn \times n$ duplication matrix such that $Y_{t+h} = \bar{S} S_{j_{t+h}} Y_{t+h}$ for any h.

Proof — See appendix.

For instance, the forecast of Y_{t+H} conditional on future shocks taking their median values can be obtained by choosing the $\{S_{j_{t+h}}\}_{h=1}^{H}$ matrices such that they select the median quantile.

Equations (8)-(9) are a generalization of (5). Relationship (5) implicitly assumes a specific sequence of shocks. For instance, the first element of (5) is the θ_1 quantile associated with the first dependent variable of all the future and cross-sectional θ_1 quantiles of the dependent variables. This corresponds to the first element of (8)-(9) when the sequence $\{S_{j_{t+h}}\}_{h=1}^{H}$ selects the following shocks $\{\epsilon^{\theta_1}_{1,t+1}, \ldots, \epsilon^{\theta_1}_{n,t+1}, \ldots, \epsilon^{\theta_1}_{1,t+H}, \ldots, \epsilon^{\theta_1}_{n,t+H}\}$ to be set to zero. Equations (8)-(9) allow one to forecast any quantile of *any* future and cross-sectional quantile.

It is also possible to rewrite the impulse response function in terms of (8)-(9).

Theorem 6 ***(Generic Quantile Impulse Response Function)*** — *Suppose that at time t the shock $\epsilon^{\theta_j}_{it} = 0$ had realized, for some $j \in \{1, \ldots, p\}$, resulting in the realization $\ddot{Y}_t$, instead of Y_t. The change in forecast, given this shock, is:*

$$\hat{Y}_{t+1}|\ddot{Y}_t - \hat{Y}_{t+1} = \bar{B}^{\theta}_{t+1} A^{\theta}_1 (\ddot{Y}_t - Y_t) \tag{10}$$

$$\hat{Y}_{t+H}|\ddot{Y}_t - \hat{Y}_{t+H} = \bar{B}^{\theta}_{t+H} A^{\theta}_1 (\hat{Y}_{t+H-1}|\ddot{Y}_t - \hat{Y}_{t+H-1}) \qquad \textit{for } H \geq 2 \tag{11}$$

Proof — See appendix.

The greater generality and flexibility of (8) provides the natural environment to perform stress testing exercises. A policy maker interested in how the endogenous variables react to a given stress scenario can first define the scenario by choosing a series of future tail (say, 10% or 1%) quantiles of interest, and then obtain the forecast of the endogenous variables conditional on the chosen scenario.

A corollary of the previous results is that it is straightforward to compute average step ahead forecasts from the QVAR model.

Corollary 1 ***(Average step ahead forecast)*** — *Consider the average H-step ahead values of the dependent variables at time t, that is:*

$$Y_{t,H} \equiv H^{-1} \sum_{h=1}^{H} Y_{t+h} \tag{12}$$

The forecast, conditional on the scenario identified by the sequence of matrices $\{S_{j_{t+h}}\}_{h=1}^{H}$, is:

$$\hat{Y}_{t,H} \equiv H^{-1} \sum_{h=1}^{H} \hat{Y}_{t+h}$$

where $\hat{Y}_{t+h}$ is defined in (8)-(9).

Proof — See appendix.

2.3 General quantile VAR(q) model

Model (4) can be generalized to any desired lag order q using its companion form. Define the npq vectors $\bar{\omega} \equiv [(\omega^{\theta})', 0', \ldots, 0']'$, $\bar{Y}_{t+1} \equiv [Y'_{t+1}, Y'_{t}, \ldots, Y'_{t-q+2}]'$, $\varepsilon_{t+1} \equiv [(\epsilon^{\theta}_{t+1})', 0', \ldots, 0']'$, and the $(npq \times npq)$ matrices

$$A^0 = \begin{bmatrix} A_0^{\theta}, & \mathbf{0}, & \ldots, & \mathbf{0} \\ \mathbf{0}, & \mathbf{0}, & \ldots, & \mathbf{0} \\ \vdots & & \ddots & \\ \mathbf{0}, & \mathbf{0}, & \ldots, & \mathbf{0} \end{bmatrix} \text{ and } A^1 = \begin{bmatrix} A_1^{\theta}, & A_2^{\theta}, & \ldots, & A_q^{\theta} \\ I_{np}, & \mathbf{0}, & \ldots, & \mathbf{0} \\ \vdots & & \ddots & \\ \mathbf{0}, & \ldots, & I_{np}, & \mathbf{0} \end{bmatrix}.$$

Then the companion form of the VAR(q) model is:

$$\bar{Y}_{t+1} = \bar{\omega} + A^0 \bar{Y}_{t+1} + A^1 \bar{Y}_t + \varepsilon_{t+1} \tag{13}$$

All the results of the previous sections extend to model (13).

2.4 Estimation and Asymptotics

The recursive QVAR model (4) can be estimated using the framework developed by White et al. (2015). Let $q_t^{\theta}(\beta) \equiv \omega^{\theta} + A_0^{\theta} Y_t + A_1^{\theta} Y_{t-1}$ and $q_{it}^{\theta_j}(\beta)$ the j^{th} quantile of the i^{th} variable of the vector $q_t^{\theta}(\beta)$, where we have made explicit the dependence on β, the vector containing all the unknown parameters in ω^{θ}, A_0^{θ}, and A_1^{θ}. Define the quasi-maximum likelihood estimator $\hat{\beta}$ as the solution of the optimization problem:

$$\hat{\beta} = \arg\min_{\beta} T^{-1} \sum_{t=1}^{T} \left\{ \sum_{i=1}^{n} \sum_{j=1}^{p} \rho_{\theta} \left(\tilde{Y}_{it} - q_{it}^{\theta_j}(\beta) \right) \right\}, \tag{14}$$

where $\rho_{\theta}(u) \equiv u(\theta - I(u < 0))$ is the standard check function of quantile regressions. The asymptotic distribution of the regression quantile estimator is provided by White et al. (2015), which we report here for convenience.

Theorem 7 ***(White et al., 2015)*** — *Under the assumptions of theorems 1 and 2 of White et al. (2015), $\hat{\beta}$ is consistent and asymptotically normally* *distributed. The asymptotic distribution is:*

$$\sqrt{T}(\hat{\beta} - \beta^*) \xrightarrow{d} N(0, Q^{-1}VQ^{-1}) \quad (15)$$

where

$$
\begin{aligned}
Q &\equiv \sum_{i=1}^{n}\sum_{j=1}^{p} E[f_{it}^{\theta_j}(0)\nabla q_{it}^{\theta_j}(\beta^*)\nabla' q_{it}^{\theta_j}(\beta^*)] \\
V &\equiv E[\eta_t \eta_t'] \\
\eta_t &\equiv \sum_{i=1}^{n}\sum_{j=1}^{p} \nabla q_{it}^{\theta_j}(\beta^*)\psi^{\theta_j}(\epsilon_{it}^{\theta_j}) \\
\psi^{\theta_j}(\epsilon_{it}^{\theta_j}) &\equiv \theta_j - I(\epsilon_{it}^{\theta_j} \leq 0) \\
\epsilon_{it}^{\theta_j} &\equiv \tilde{Y}_{it} - q_{it}^{\theta_j}(\beta^*)
\end{aligned}
$$

and $f_{it}^{\theta_j}(0)$ *is the conditional density function of* $\epsilon_{it}^{\theta_j}$ *evaluated at 0.*

The asymptotic variance-covariance matrix can be consistently estimated as suggested in theorems 3 and 4 of White et al. (2015), or using bootstrap based methods in the spirit of Buchinsky (1995).[2]

The following corollary derives the standard errors of the forecasts.

Corollary 2 ***(Forecast standard errors)*** — *Let* $Y_{T+H}(\hat{\beta}) \equiv \hat{Y}_{T+H}$ *the forecast* (8)-(9)*, where it has been made explicit the dependence on the model*

[2] Modern statistical softwares contain packages for regression quantile estimation and inference. This paper uses the interior point algorithm discussed by Koenker and Park (1996).

parameters β. Then:

$$\sqrt{T}(Y_{T+H}(\hat{\beta}) - Y_{T+H}(\beta^*)) \xrightarrow{d} N(0, \Phi(\beta^*)Q^{-1}VQ^{-1}\Phi'(\beta^*)) \quad (16)$$

where $\Phi(\beta^*) \equiv \partial Y_{T+H}(\beta^*)/\partial\beta'$.

Proof — See appendix.

The standard errors associated with the impulse response function (10)-(11) can be computed in a similar fashion.

3 Stress testing the euro area economy

We apply the methodology developed in the previous section to model the interaction between real and financial variables in Europe. We study the interrelationship between the euro area industrial production growth ($\tilde{Y}_{1t}$) and the composite indicator of systemic stress in the financial system (CISS, $\tilde{Y}_{2t}$) of Hollo, Kremer and Lo Duca (2012). We perform three exercises. First, we estimate short term euro area growth at risk (defined as the 10% quantile of $\tilde{Y}_{1t}$), as a function of financial conditions. Second, we forecast euro area growth under a severe stress scenario, where both the real and financial parts of the euro area economy are hit by a sequence of six consecutive tail shocks. Third, we ask whether the quantile VAR methodology could have been helpful in detecting vulnerabilities in the months preceding Lehman Brothers' default.

3.1 Euro area growth at risk

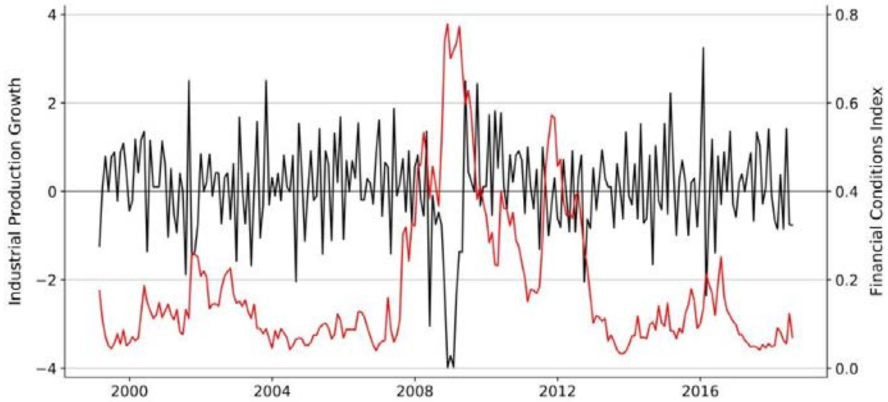
Adrian et al. (2019) have shown that there are substantial asymmetries in the relationship between the US real GDP growth and financial conditions. In particular, they find that the estimated lower quantiles of the distribution of future GDP growth are significantly affected by financial conditions, while the upper quantiles appear to be more stable over time. The quantile model specification of Adrian et al. (2019) is the following:

$$\tilde{Y}_{1,t+1} = \omega_1^\theta + a_{11}^\theta \tilde{Y}_{1,t} + a_{12}^\theta \tilde{Y}_{2t} + \epsilon_{t+1}^\theta \tag{17}$$

They estimate this model for $\theta \in \{0.05, 0.25, 0.75, .95\}$. This corresponds to the first line of model (4). An obvious drawback of neglecting the second line of the quantile VAR model is that iterated multiperiod ahead forecasts become impossible. In fact, for the four quarters ahead analysis, they have to resort to direct estimation, whereby they quantile regress the four quarter ahead GDP directly on current GDP and financial conditions. Our framework, instead, allows us to estimate the model at the highest possible frequency and still to study the forecasting properties of the system as well as to test the presence of any feedback effect. While there are merits in direct forecast approaches (see, for instance, McCracken and McGillicuddy, 2018), iterated forecasts allow for richer scenario analyses.

We start by reporting in figure 1 the monthly time series of industrial production and CISS in the euro area from January 1999 until July 2018.

Figure 1: Real and financial variables



Note: Time series evolution of euro area industrial production (dash line) and Composite Indicator of Systemic Stress (CISS, continuous line). Monthly data, 1999:01-2018:07. Source: ECB.

The data is downloaded from the Statistical Data Warehouse database of the ECB.[3] A cursory view at the plot reveals a clear negative correlation between the two time series, especially during the Great Financial Crisis.

Next, we estimate the quantile VAR model (4):

$$\tilde{Y}_{1,t+1} = \omega_1^\theta \qquad\qquad\quad + a_{11}^\theta \tilde{Y}_{1t} + a_{12}^\theta \tilde{Y}_{2t} + \epsilon_{1,t+1}^\theta \tag{18}$$

$$\tilde{Y}_{2,t+1} = \omega_2^\theta + a_0^\theta \tilde{Y}_{1,t+1} + a_{21}^\theta \tilde{Y}_{1t} + a_{22}^\theta \tilde{Y}_{2t} + \epsilon_{2,t+1}^\theta \tag{19}$$

By ordering CISS after industrial production, we impose the structural identification assumption that financial variables can react contemporaneously to real variables, but real variables react to financial developments

[3]Available at https://sdw.ecb.de/home.do.

only with a lag. This corresponds to a Choleski identification where shocks to real economic variables can have an immediate impact on financial variables, while shocks to financial variables are allowed to affect real variables only with a lag. Given the speed at which financial markets react to news, this seems like a reasonable assumption.

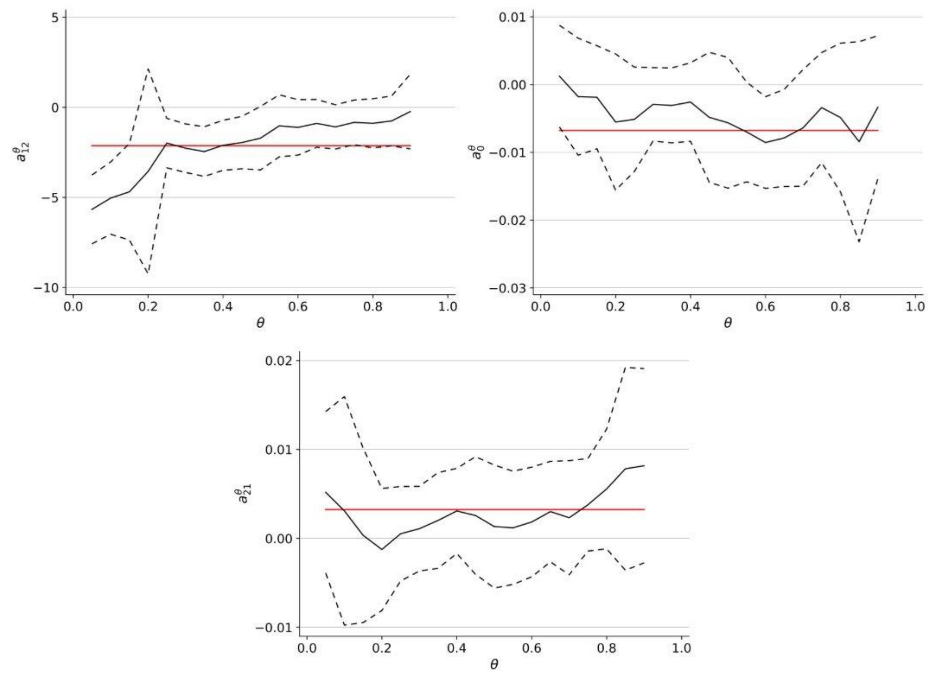
The interaction between real and financial variables can be tested by checking whether the off-diagonal coefficients are statistically different from zero:

$$H_0 : a_{12}^{\theta} = 0, \quad H_0 : a_0^{\theta} = 0, \quad H_0 : a_{21}^{\theta} = 0 \tag{20}$$

Figure 2 reports the estimated quantile coefficients $a_{12}^{\theta}, a_0^{\theta}, a_{21}^{\theta}$ for $\theta \in \{0.05, 0, 10, 0, 15, \ldots, 0.95\}$, together with the OLS estimate. We observe the presence of substantial asymmetries, especially in the a_{12}^{θ} coefficient, which cannot be detected with standard OLS models. The coefficient estimates of a_{12}^{θ} are consistent with the findings of Adrian et al. (2019), whereby financial conditions significantly affect the left tail of the distribution of industrial production, but not the right tail.

In the top panel of figure 3, we show that the impact of financial conditions is not only statistically significant, but also economically relevant. The figure reports the 10% and 90% quantiles of industrial production. It reveals that worsening of financial conditions impacts the left tail by about two percentage points. The middle line represents the estimated conditional expectation of industrial production according to a standard OLS VAR model. Notice

Figure 2: Testing interactions between real and financial variables



Note: Estimated coefficients of the off diagonal elements at different θ quantiles, with 90% confidence intervals. The flat line represent the OLS estimate.

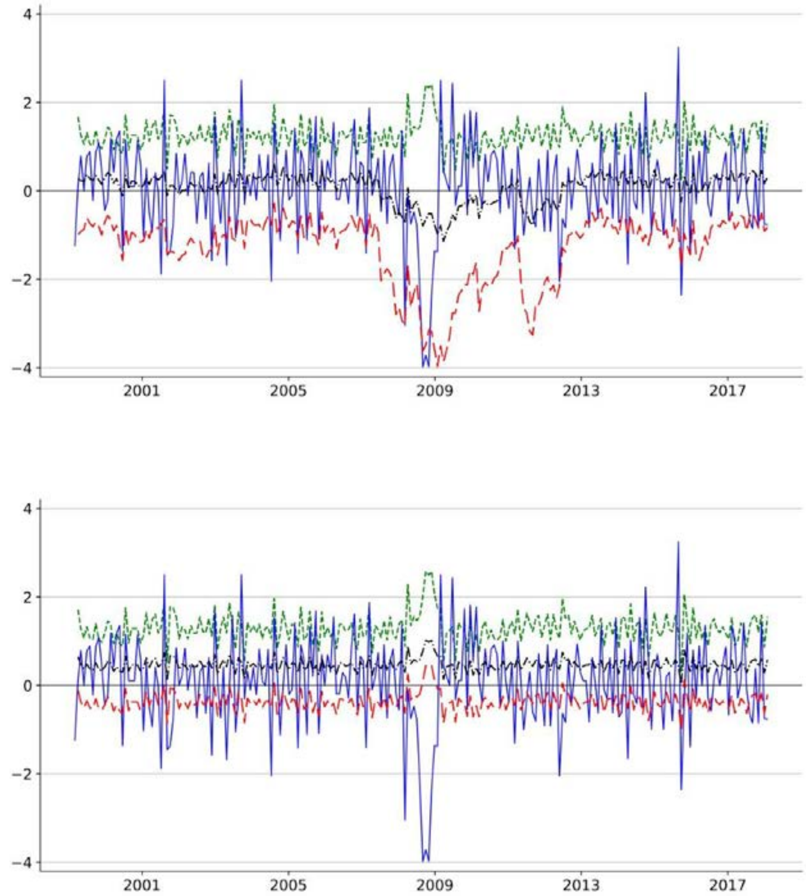
that the impact of the financial crisis is much more muted relative to the one obtained with the 10% quantile. For comparison, in the bottom panel of figure 3 we report the same time series quantile estimates of industrial production where the off-diagonal coefficient a_{12}^{θ} has been set to zero.

In figure 4 we compute a three dimensional quantile impulse response function corresponding to (6), which studies how different quantiles of industrial production react to a shock to CISS. The thought experiment is the following: How different the various quantiles would have been if we had observed a different realization in the financial conditions of the euro area economy? The change in quantile forecasts is measured along the vertical axis (QIRF), while the horizontal plane contains the different quantiles (θ) and time horizons (h). We continue to notice substantial asymmetric impacts in different parts of the distribution, but the chart now reveals that these asymmetries disappear after around 24 periods, which corresponds to two years. This analysis highlights the advantage of our framework. It is an internally consistent fully dynamic model of the real and financial variables of the euro area economy, which allows us to study the propagation of shocks across the different parts of the distribution and through time.

3.2 Forecasting growth under stress scenarios

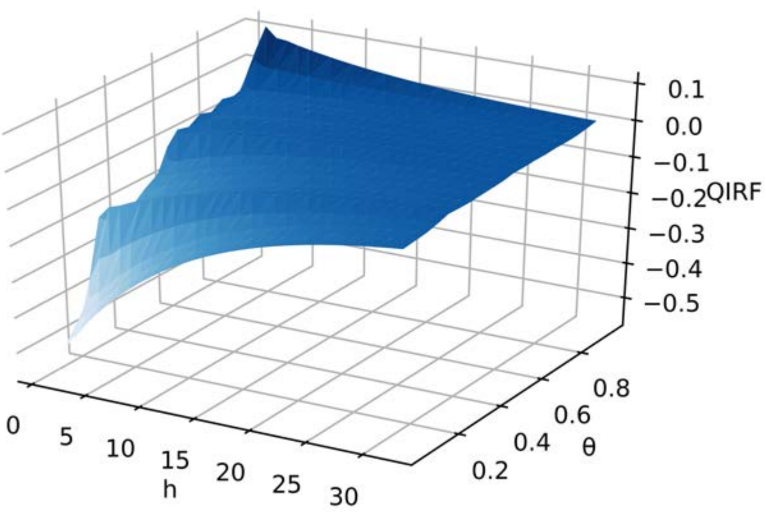
In figure 5, we report the forecast distribution of industrial production several months ahead, conditional on the future endogenous variables being hit by different quantile realizations. Each dotted line corresponds to alternative

Figure 3: Euro area growth at risk



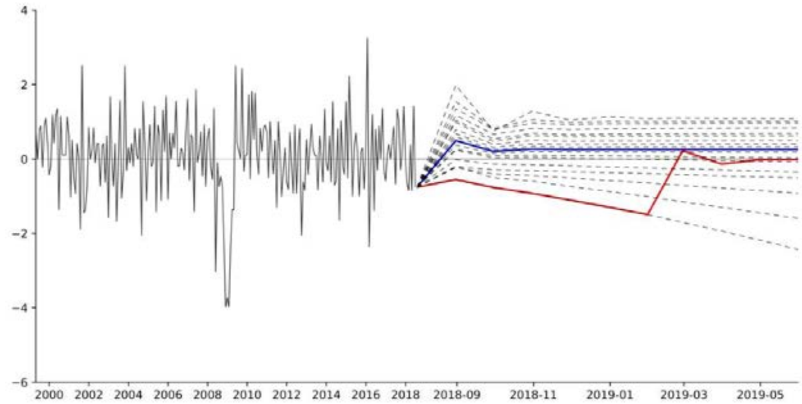
Note: Time series estimates of the 10% and 90% quantiles of euro area industrial production, together with the mean estimate according to a standard OLS VAR. The top panel represents the unrestricted estimates, the bottom panel restricts the off-diagonal coefficients to be zero.

Figure 4: Quantile impulse response function for the euro area industrial production



Note: The figure reports how a shock to the financial variable would affect the estimates of the different quantiles of euro area industrial production at different time horizons.

Figure 5: Forecasting and stress testing the real and financial variables in the euro area



Note: The figure reports the time series of industrial production for the euro area together with the forecasts associated with different scenarios. The path highlighted in blue corresponds to a scenario where both the real and financial variables are hit by a sequence of median shocks. The path highlighted in red corresponds to the stress scenario where the financial variable is hit by a 90% shock and the real variable by a 10% shock for six consecutive months, followed by median shocks.

specifications for the sequence of $\{S_{j_{T+h}}\}_{h=1}^{10}$ matrices in (8)-(9). The various dots at each point in time can be thought as possible realizations from the distribution of the future random variables.

We have highlighted two specific scenarios. The one in blue corresponds to a situation where the sequence of future random variables are set to their median values. This roughly corresponds to the results that one would obtain from a standard OLS VAR analysis. Our framework, however, allows us also to create arbitrary stress scenarios and to assess their impact. In the same figure, we have highlighted in red the forecast of the system associated with the following stress testing exercise. We assume that the euro area economy

Figure 6: Evolution of industrial production under alternative scenarios

Note: The figure reports the historical time series of industrial production together with its projected levels as of July 2018 under the stress scenario (red line) and median scenario (blue line). The stress scenario is defined as in figure 5 as a sequence of six monthly 90% financial and 10% real shocks, followed by a sequence of median shocks.

is hit by a series of six consecutive 90% quantile realizations to its financial system and 10% quantile realizations to its real economy. This can be seen by the fact that the red line initially follows the trajectory of the bottom dotted line, which traces the forecasts associated with consecutive 90% and 10% quantile realizations. After that, we assume that the system is hit by a series of median shocks, reverting to normal functioning. We see that industrial production contracts by a maximum of around 2%.

Figure 6 reports the implication of the scenarios of figure 5 in levels of industrial production, by cumulating the monthly growth forecasts. Notice that the chosen stress scenario implies an overall contraction in industrial production of more than 10% over 6 months, a contraction falling somewhere in between the one experienced during the financial crisis in 2008-2009 and that of the euro area sovereign debt crisis in 2012. Charts of this type can be used by policy makers to calibrate the severity of the stress test according to their own preferences.

3.3 Counterfactual scenario analysis before Lehman Brothers' default

One year after the collapse of Lehman Brothers, Queen Elizabeth II famously asked: *Why did nobody notice it?* From the perspective of the methodology developed in this paper, predicting a crisis and its severity is like predicting that a certain sequence of adverse quantile realizations will hit the system. This is impossible, to the same extent as it is impossible to predict which number of the roulette will come out in the next round. It is possible, however, to use the quantile VAR methodology to measure if an economy is resilient to alternative stress scenarios.

We estimate the model (18)-(19) using data only up to August 2008. For given parameter estimates, we use the system to forecast industrial production six months ahead under the following three scenarios:

1. **Good financial scenario**: sequence of six 10% quantile realizations for both industrial production and CISS.

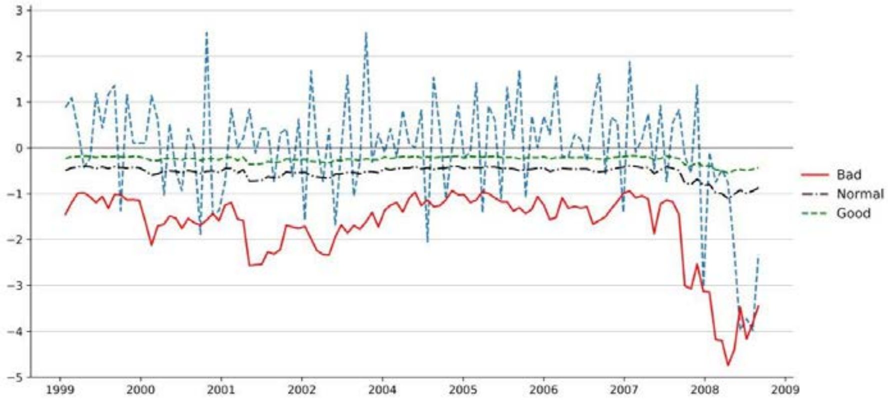
2. **Normal financial scenario**: sequence of six 10% quantile realizations for industrial production and median realizations for CISS.

3. **Bad financial scenario**: sequence of six 10% and 90% quantile realizations for industrial production and CISS, respectively.

The estimates are reported in figure 7. It is evident that the good and normal financial scenarios were posing little risks to the euro area economy, since even after a sequence of six negative shocks to the real economy, growth at risk was quite contained. It is only under the combination of adverse real and financial shocks that growth at risk is significantly affected. In fact, already in mid 2007, growth at risk under this adverse scenario had reached unprecedented magnitudes for the euro area, from an historical perspective. The large growth at risk under the bad financial scenario reveals the presence of a fat left tail in the distribution of the euro area industrial production, which would go unnoticed by simply estimating the 5% growth at risk using direct estimation techniques.

More generally, such counterfactual exercises are not feasible with the direct forecast approach. By directly quantile regressing industrial production six months ahead against current real and financial conditions, one implicitly imposes that the system evolves according to some average scenario during the intervening six months. While this may be a reasonable assumption if one is interested in modeling the conditional mean of the endogenous variables, it seems like an undesirable constraint to impose when modeling their tail behavior. Notice, however, that if one is interested in such unconditional scenario, this can be recovered from the empirical distribution obtained by simulating the quantile VAR under all alternative quantile scenarios (simi-

Figure 7: Growth at risk under alternative scenarios as of August 2008



Note: six month ahead forecast of euro area industrial production under three alternative scenarios. The good, normal and bad scenarios are defined by a sequence of six consecutive benign, normal and adverse financial shocks. The parameter of the quantile VAR are estimated using only observations up to August 2008.

larly to all the possible dotted lines of figure 5) and then choosing the desired empirical quantile forecast.

4 Conclusion

We have developed a quantile VAR model and used it to forecast and stress test the interaction between real and financial variables in the euro area. Unlike OLS VAR, quantile VAR models each quantile of the distribution. This provides the natural modeling environment to design particular stress scenarios and test the impact that they have on the economy. A stress scenario is just a sequence of tail quantile shocks, which can be chosen arbitrarily by the policy maker or calibrated to mimic previous crisis episodes. We find the presence of strong asymmetries in the transmission of financial shocks in the euro area, with negative financial shocks being particularly harmful when coupled with negative real shocks. By modelling the average interaction between the random variables, OLS VAR models miss most of these detrimental interactions.

Appendix — Proofs

Proof of Theorem 1 (VAR forecast) — We report here a variant of this proof, for comparison with the proof of the QVAR forecast. Rewrite the structural vector autoregressive model (1) in reduced form:

$$\tilde{Y}_{t+1} = \mu_t + (I_n - A_0)^{-1}\epsilon_{t+1}$$

where $\mu_t \equiv (I_n - A_0)^{-1}\omega + (I_n - A_0)^{-1}A_1\tilde{Y}_t$. The expected value of the process (1) at time $t + H$ can be rewritten as:

$$\begin{aligned} E_{t+H}(\tilde{Y}_{t+H+1}) &= \mu_{t+H} \\ &= \nu + B\tilde{Y}_{t+H} \end{aligned}$$

Solving the system backwards in terms of the structural shocks $\{\epsilon_{t+h}\}_{h=1}^{H}$, for $H \geq 1$:

$$\mu_{t+H} = \sum_{h=0}^{H} B^h \nu + B^{H+1} \tilde{Y}_t + \sum_{h=1}^{H} B^{H-h+1} (I_n - A_0)^{-1} \epsilon_{t+h}$$

Since μ_{t+H} depends on future shocks, it is a random variable. Computing the expectation of its future expectations, we obtain the desired result:

$$E_t(\cdots E_{t+H-1}(\mu_{t+H})) = \sum_{h=0}^{H} B^h \nu + B^{H+1} \tilde{Y}_t$$

□

Proof of Theorem 2 (Mean Impulse Response Function) — Rewrite the forecast in terms of structural shocks:

$$\begin{aligned} E_t(\tilde{Y}_{t+H}) &= \sum_{h=0}^{H} B^h \nu + B^{H+1} \tilde{Y}_t \\ &= \sum_{h=0}^{H} B^h \nu + B^{H+1} (\mu_{t-1} + (I_n - A_0)^{-1} \epsilon_t) \end{aligned}$$

By taking the derivative with respect to ϵ_t, the result follows. □

Proof of Theorem 3 (Quantile VAR forecast) — Rewrite the θ quantile of process (4) in reduced form:

$$Y_{t+1} = q_t^\theta + (I_{np} - A_0^\theta)^{-1} \epsilon_{t+1}^\theta$$

where $q_t^\theta = (I_{np} - A_0^\theta)^{-1} \omega^\theta + (I_{np} - A_0^\theta)^{-1} A_1^\theta Y_t$ and I_{np} is the identity matrix of dimension np.

The quantile at time $t+H$, given the information available at $t+H$, is:

$$Q^{\theta}_{t+H}(Y_{t+H+1}) = q^{\theta}_{t+H} + Q^{\theta}_{t+H}((I_{np} - A^{\theta}_0)^{-1}\epsilon^{\theta}_{t+H+1})$$

where $Q^{\theta}_{t+H}(\cdot)$ is the cross sectional quantile operator of definition 2. Assume first that θ is a scalar and let $A \equiv (I_{np} - A^{\theta}_0)^{-1}$, with typical entry a_{ij} and row a_i, $i, j = 1, \dots n$. Since A^{θ}_0 is lower triangular with zeros along the main diagonal, also A is lower triangular. The last element of the vector $A\epsilon^{\theta}_{t+H+1}$ can therefore be written as $a_n\epsilon^{\theta}_{t+H+1} = a_{n1}\epsilon^{\theta}_{1,t+H+1} + \dots + a_{n,n-1}\epsilon^{\theta}_{n-1,t+H+1} + \epsilon^{\theta}_{n,t+H+1}$. By definition of process (4), the quantile of this expression conditional on $\Omega_{n,t+H}$ is $Q^{\theta}_{n,t+H}(a_n\epsilon^{\theta}_{t+H+1}) = a_{n1}\epsilon^{\theta}_{1,t+H+1} + \dots + a_{n,n-1}\epsilon^{\theta}_{n-1,t+H+1}$, because $Q^{\theta}_{n,t+H}(\epsilon^{\theta}_{n,t+H+1}) = 0$ and the other elements belong to the information set $\Omega_{n,t+H}$. The quantile of this expression, in turn, conditional on the information set $\Omega_{n-1,t+H}$, that is, excluding the last contemporaneous variable, is $Q^{\theta}_{n-1,t+H}(Q^{\theta}_{n,t+H}(a_n\ \epsilon^{\theta}_{t+H+1})) = a_{n1}\epsilon^{\theta}_{1,t+H+1} + \dots + a_{n,n-2}\epsilon^{\theta}_{n-2,t+H+1}$.[4] Continuing taking the conditional quantile for all the cross section, we get $Q^{\theta}_{1,t+H}(\dots Q^{\theta}_{n,t+H}(a_n\epsilon^{\theta}_{t+H+1})) = 0$. Repeating the same reasoning for all the rows of the matrix A, we get in the more concise notation of definition 2:

$$Q^{\theta}_{t+H}((I_{np} - A^{\theta}_0)^{-1}\epsilon^{\theta}_{t+H+1}) = 0$$

[4] Here we have a slight abuse of notation, as $P(a_{n,n-1}\epsilon^{\theta}_{n-1,t+H+1} < 0|\Omega_{n-1,t+H}) = 1-\theta$ when $a_{n,n-1} < 0$, and so it becomes the $(1-\theta)$ quantile.

The case when θ is a vector can be proven in a similar fashion, by applying the same reasoning to each individual quantile block of the np-vector $A\epsilon^{\theta}_{t+H+1}$.

Therefore:

$$
\begin{aligned}
Q^{\theta}_{t+H}(Y_{t+H+1}) &= q^{\theta}_{t+H} \\
&= \nu^{\theta} + B^{\theta} Y_{t+H} \\
&= \nu^{\theta} + B^{\theta} q^{\theta}_{t+H-1} + B^{\theta}(I_{np} - A^{\theta}_{0})^{-1}\epsilon^{\theta}_{t+H}
\end{aligned}
$$

where $\nu^{\theta} \equiv (I_{np} - A^{\theta}_{0})^{-1}\omega^{\theta}$ and $B^{\theta} \equiv (I_{np} - A^{\theta}_{0})^{-1}A^{\theta}_{1}$.

For $H \geq 1$, recursive substitution gives:

$$
q^{\theta}_{t+H} = \sum_{h=0}^{H}(B^{\theta})^{h}\nu^{\theta} + (B^{\theta})^{H+1}Y_{t} + \sum_{h=1}^{H}(B^{\theta})^{H-h+1}(I_{np} - A^{\theta}_{0})^{-1}\epsilon^{\theta}_{t+h}
$$

Notice that like μ_{t+H} also q^{θ}_{t+H} is a random vector at time t, as it depends on the vector of future structural shocks ϵ^{θ}_{t+h}. Applying recursions over time similar to those outlined above for the cross section gives the θ quantile of future θ quantiles:

$$
Q^{\theta}_{t}(\cdots Q^{\theta}_{t+H}(Y_{t+H+1})) = \sum_{h=0}^{H}(B^{\theta})^{h}\nu^{\theta} + (B^{\theta})^{H+1}Y_{t}
$$

because $Q^{\theta}_{t+h-1}((B^{\theta})^{H-h+1}(I_{np} - A^{\theta}_{0})^{-1}\epsilon^{\theta}_{t+h}) = 0$ for all h. $\square$

Proof of Theorem 4 (Quantile Impulse Response Function) — Rewrite the quantile forecast in terms of structural shocks:

$$Q_t^\theta(\cdots Q_{t+H}^\theta(Y_{t+H+1})) = \sum_{h=0}^{H}(B^\theta)^h\nu^\theta + (B^\theta)^{H+1}Y_t$$
$$= \sum_{h=0}^{H}(B^\theta)^h\nu^\theta + (B^\theta)^{H+1}(q_{t-1}^\theta + (I_{np} - A_0^\theta)^{-1}\epsilon_t^\theta)$$

By taking the derivative with respect to ϵ_t^θ, the result follows. □

Proof of Theorem 5 (Generic quantile VAR forecast) — By (4), the forecast of $\tilde{Y}_{t+1}$, conditional on setting the residuals identified by the matrix $S_{j_{t+1}}$ to zero, is:

$$\underset{(n\times 1)}{\hat{Y}_{t+1}}|S_{j_{t+1}} = S_{j_{t+1}}Y_{t+1}$$
$$= S_{j_{t+1}}(\omega^\theta + A_0^\theta Y_{t+1} + A_1^\theta Y_t)$$
$$= S_{j_{t+1}}\omega^\theta + S_{j_{t+1}}A_0^\theta \bar{S} S_{j_{t+1}}Y_{t+1} + S_{j_{t+1}}A_1^\theta Y_t$$

where we have made use of the equality $\bar{S}S_{j_{t+1}}Y_{t+1} = Y_{t+1}$. Solving for $S_{j_{t+1}}Y_{t+1}$ and iterating the equation forward, for any given sequence $\{S_{j_{t+h}}\}_{h=1}^{H}$, we obtain the result. □

Proof of Theorem 6 (Generic Quantile Impulse Response Function) — The result follows immediately from equation (8). □

Proof of Corollary 1 (Average step ahead forecast) — The result follows from (8) and from observing that each forecast is not random. □

Proof of Corollary 2 (Forecast standard errors) — Consider the mean value expansion $Y_{T+H}(\hat{\beta}) = Y_{T+H}(\beta^*) + \Phi(\bar{\beta})(\hat{\beta} - \beta^*)$. The result follows from the asymptotic properties of $\hat{\beta}$. $\square$

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