Gaussian Mixture Approximations of Impulse Responses and the Non-Linear Effects of Monetary Shocks

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Effects of monetary policy on the real economy

- Broad consensus on the average effect of monetary policy on output
  - A monetary contraction (expansion) leads to a decline (increase) in output

How about non-linear effects of monetary policy?

1. Sign-dependence?
   - Pushing vs. pulling on a string
2. State-dependence?
   - Is pushing more effective in recessions than in expansions?

Non-linear effects of monetary policy

- No consensus in the literature
- Not surprising: standard approach to identify effect of money shock is based on linear models
  - SVAR with recursive identification scheme (Christiano, Eichenbaum and Evans, 1999)
  - SVAR with external instrument (Gertler-Karadi, 2014)

This paper

- How an economy in a given state responds to a shock of a given sign?
- New method:
  - Does not assume existence of VAR representation
  - Approximate the impulse responses with Gaussian basis functions
  - Directly parametrically estimate the structural MA representation of the system (great for prior elicitation)
  - Easily amenable to estimation of non-linear effects

Model

- $\mathbf{Y}_t$ a vector of stationary macro variables
- Structural vector MA representation of the economy

$$\mathbf{Y}_t = \sum_{k=0}^{K} \Psi_k \boldsymbol{\varepsilon}_{t-k} \tag{1}$$

  with K finite or infinite.
- With $\{\varepsilon_t\}$ the *structural* shocks affecting the economy ($E\boldsymbol{\varepsilon}_t = \mathbf{0}$ and $E\boldsymbol{\varepsilon}_t\boldsymbol{\varepsilon}_t' = \mathbf{I}$)
- We assume throughout that Ψ_0 is lower trianglular
- If $\Psi(L)$ invertible, (1) can be estimated from a VAR on $\mathbf{Y}_t$

Approximating IRFs with mixtures of Gaussians

- Intuition: IRFs often monotonic or hump-shaped
- Idea: Use Gaussian basis functions to approximate (parametrize) the impulse response functions

Approximating IRFs with mixtures of Gaussians

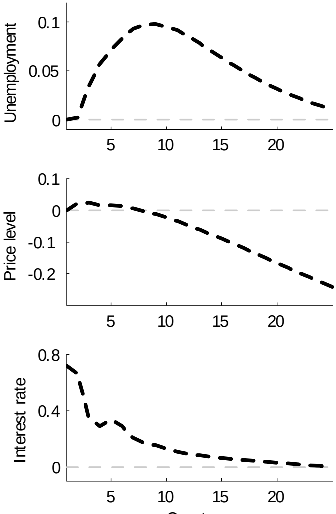
- Approximate IRF $\psi(k)$ by a sum of Gaussian basis functions:

$$\psi(k) \simeq \sum_{n=1}^{N} a_n e^{-\left(\frac{k-b_n}{c_n}\right)^2}$$

- With N Gaussians, we have a GMA(N)
- In practice, only a very small number of Gaussian basis functions are needed

Example/Motivation

- Standard SVAR with $(U, d \ln P, r)$ over 1959-2007
- IRFs to a monetary contraction

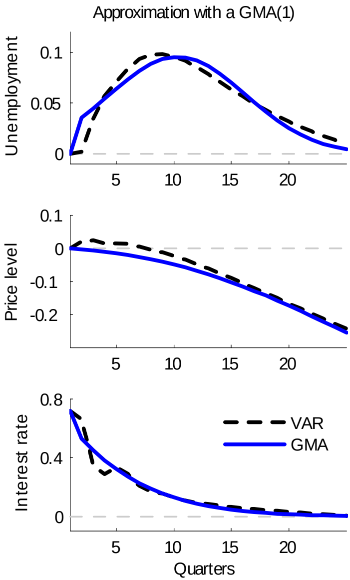


GMA(1): a one-Gaussian approximation

- Approximate the IRFs with one Gaussian function

$$\begin{cases} \psi(k) = a e^{-\left(\frac{k-b}{c}\right)^2}, \ \ \forall k > 0 \\ \psi(0) \text{ unconstrained} \end{cases}$$

Fit of a one-Gaussian approximation



GMA(1): a one-Gaussian approximation

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$$\begin{cases} \psi(k) = ae^{-\left(\frac{k-b}{c}\right)^2}, & \forall k > 0 \\ \psi(0) \text{ unconstrained} \end{cases}$$

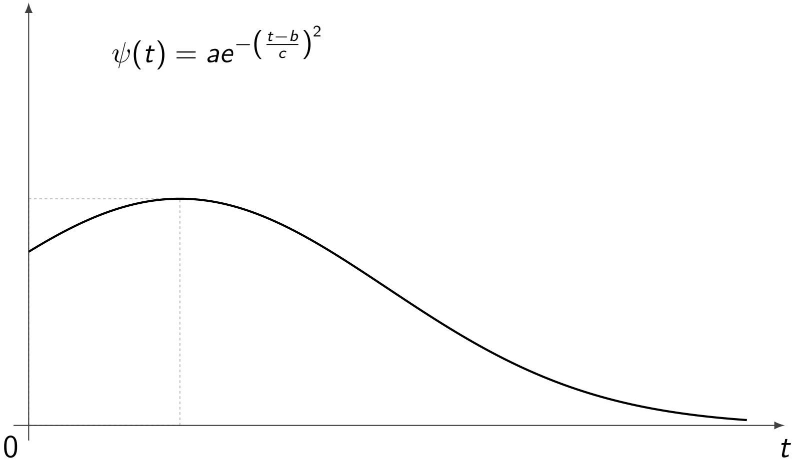
- Only 4 free parameters to capture one IRF $\{\psi(k)\}_{k=1}^{\infty}$
- For a trivariate system, nb of free parameters:
  - GMA(1): 39
  - VAR(12): 117

Advantage of a one-Gaussian parametrization

$$\psi(k) = a e^{-\left(\frac{k-b}{c}\right)^2}, \ \forall k > 0$$

- Parsimonious and yet flexible
- Can summarize IRF with 3 parameters:
  - The peak effect of the shock: a
  - The time needed to reach the peak effect: b
  - The "persistence" of the effect of the shock: c

Using one Gaussian: GMA(1)



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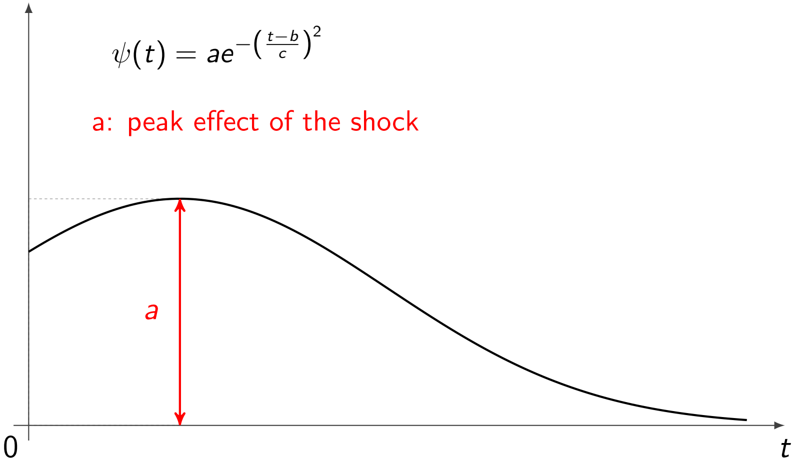

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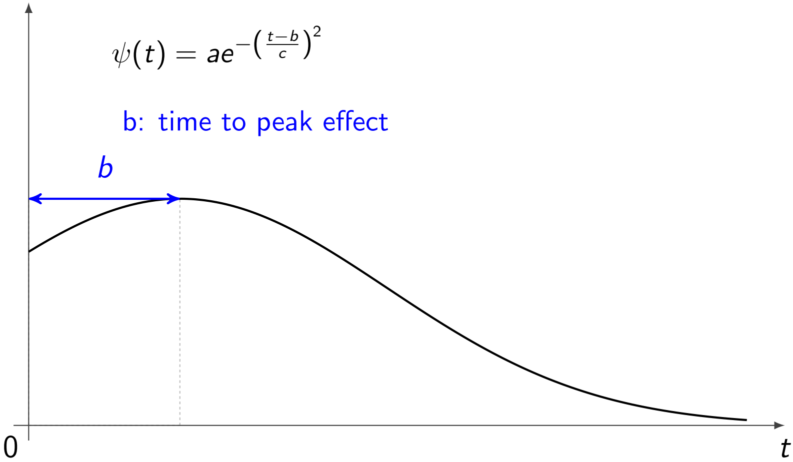

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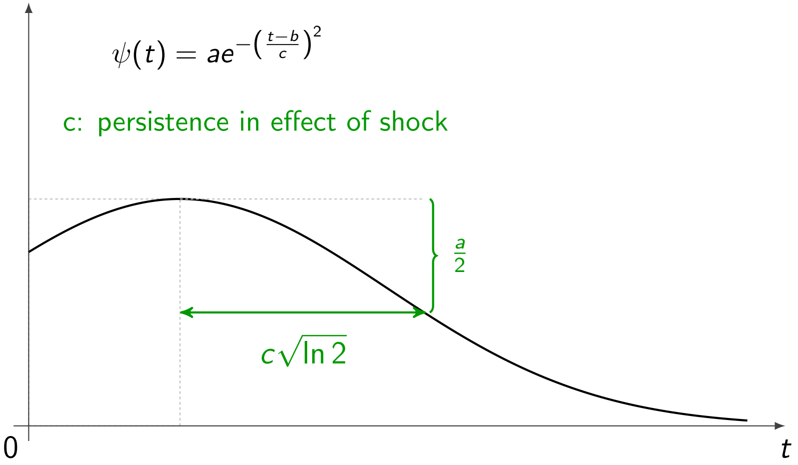

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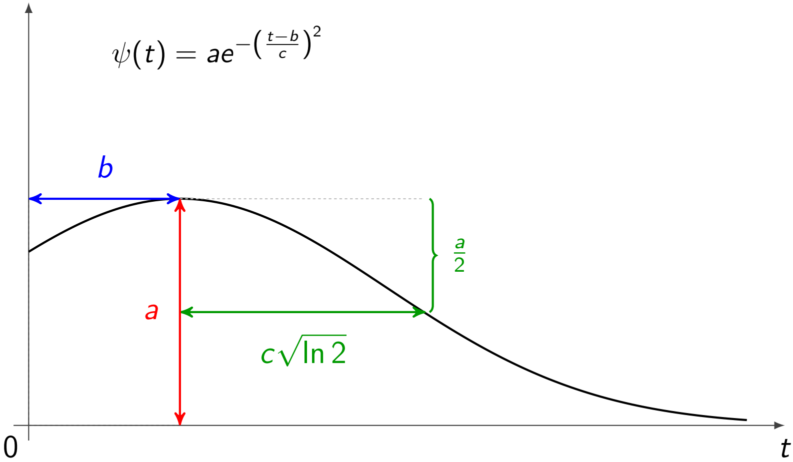
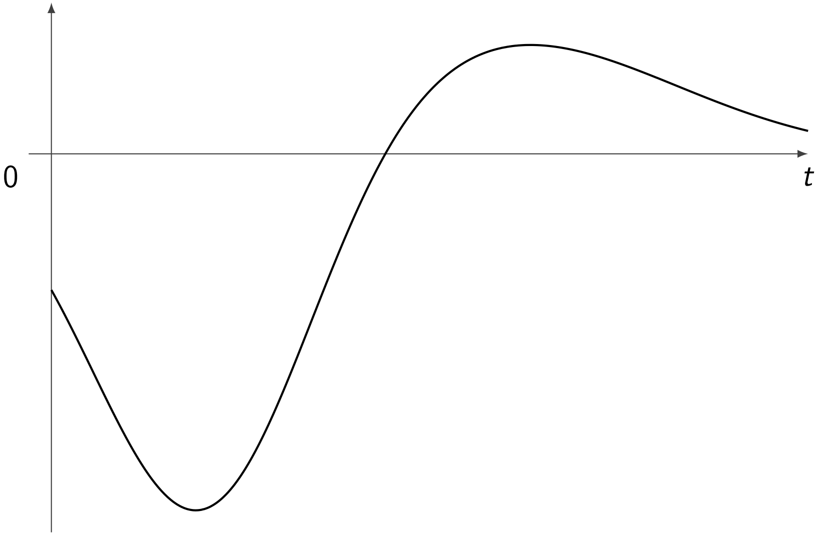


Using two Gaussians: GMA(2)



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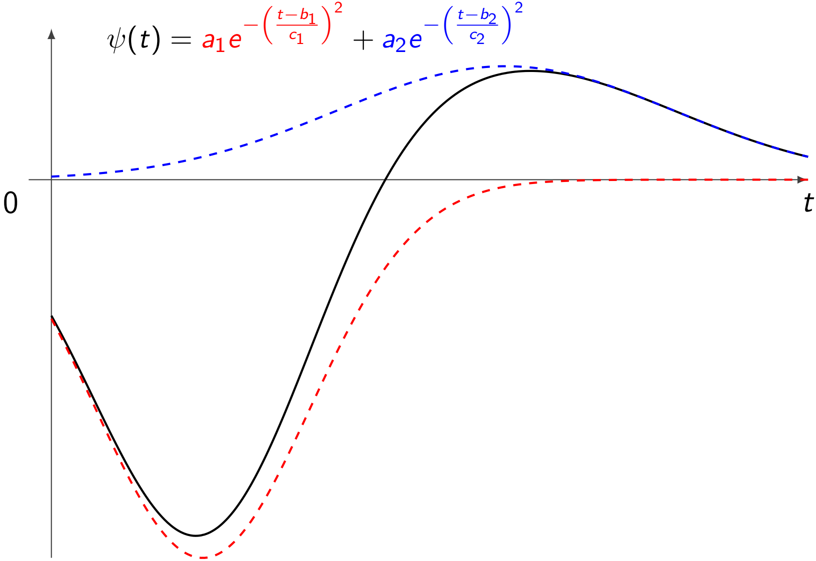
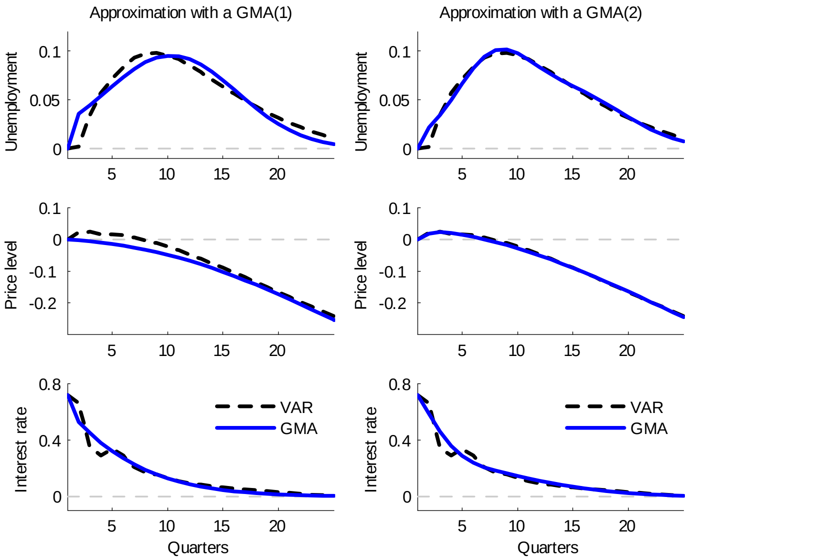
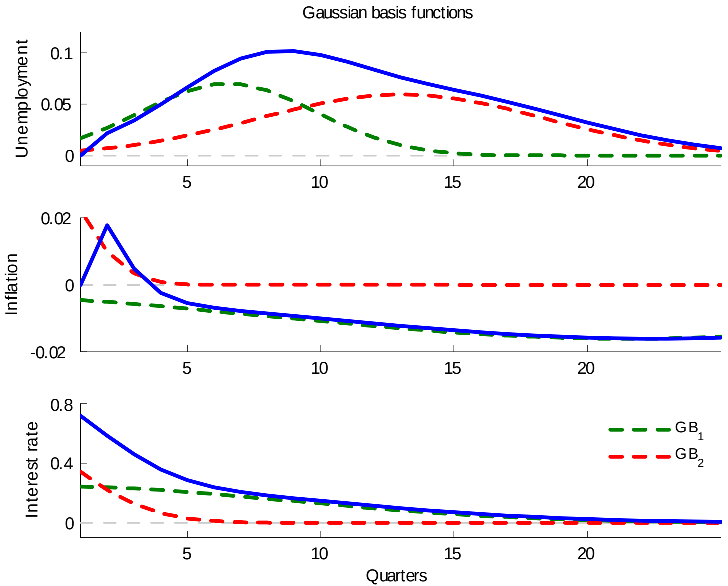


GMA(2): a two-Gaussian approximation



The Gaussian basis functions



Introducing non-linearities

- Non-linear vector MA representation of the economy

$$\mathbf{Y}_t = \sum_{k=0}^{\infty} \Psi_k(\varepsilon_{t-k}, \mathbf{Z}_{t-k})\varepsilon_{t-k}$$

- With $\{\varepsilon_t\}$ the *structural* shocks affecting the economy
- And $\mathbf{Z}_t$
  - a function of past endogenous variables $\{\mathbf{Y}_{t-j}\}_{j>0}$
  - or a function of exogenous variables

Asymmetry (1)

- Asymmetric model

$$\mathbf{Y}_t = \sum_{k=0}^{\infty} \left[\Psi_k^+ \mathbf{1}_{\varepsilon_{\mathbf{t-k}} > \mathbf{0}} \varepsilon_{t-k} + \Psi_k^- \mathbf{1}_{\varepsilon_{\mathbf{t-k}} < \mathbf{0}} \varepsilon_{t-k} \right]$$

Asymmetry (2)

- Asymmetric GMA(N)

$$\begin{cases} \psi^{+}(k) = \sum_{n=1}^{N} a_n^{+} e^{-\left(\frac{k-b_n^{+}}{c_n^{+}}\right)^2} \\ \\ \psi^{-}(k) \text{ similarly} \end{cases}$$

Asymmetry and state-dependence

- GMA(N) with asymmetry and state dependence

$$\psi^{+}(k) = (1 + \gamma^{+} z_{t-k}) \sum_{n=1}^{N} a_n^{+} e^{-\left(\frac{k-b_n^{+}}{c_n^{+}}\right)^2}$$

- Parameter γ^{+} captures state dependence at time of shock
- The state of the cycle allowed to stretch/contract the IRF, but the overall shape is fixed
- Analogy with varying-coefficient model

Bayesian estimation

- Similar to linear case
- Taking into account Jacobian of mapping between (Gaussian) structural shocks and (non-Gaussian) forecast errors

Bayesian estimation

- Construct conditional likelihood assuming Normal shocks
- Estimation routine:
  - Multiple-block Metropolis-Hastings algorithm
  - Initialize chain with GMA parameters chosen to fit VAR-based IRFs

Constructing the likelihood

- Assume Normal shocks ε_t
- Parameter vector θ
- Decompose the likelihood using

$$p(\mathbf{Y}_1, ..., \mathbf{Y}_T|\theta) = p(\mathbf{Y}_T|\mathbf{Y}_1, ..., \mathbf{Y}_{T-1}, \theta) \dots p(\mathbf{Y}_2|\mathbf{Y}_1, \theta)p(\mathbf{Y}_1|\theta)$$

- Proceed recursively with

$$p(\mathbf{Y}_{t+1}|\mathbf{Y}_1, ..., \mathbf{Y}_t, \theta) = p(\Psi_0 \boldsymbol{\varepsilon}_{t+1}|\mathbf{Y}_1, ..., \mathbf{Y}_t, \theta)$$

- And obtain shock $t+1$ from

$$\Psi_0 \boldsymbol{\varepsilon}_{t+1} = \mathbf{Y}_{t+1} - \sum_{k=0}^{K} \Psi_{k+1} \boldsymbol{\varepsilon}_{t-k}.$$

- Need Ψ_0 to be invertible, guaranteed by struct. identifying assumption (Ψ_0 lower triangular)
- Initialize recursion by setting first K shocks to zero

Other technical issues

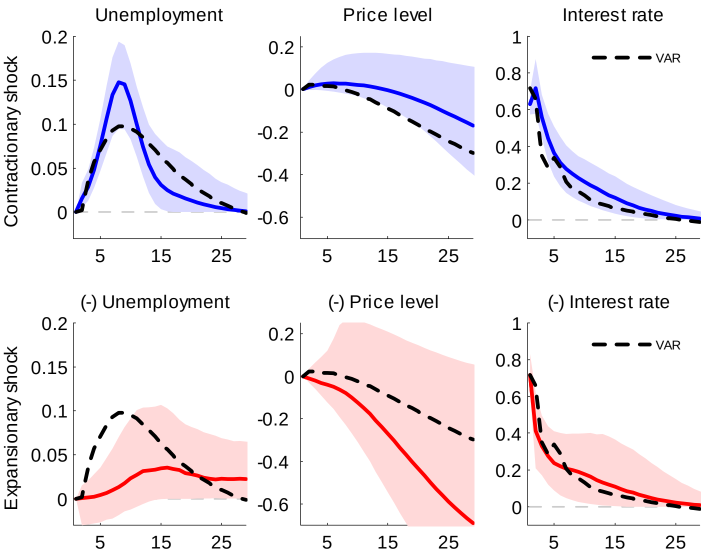
- Loose priors centered on VAR-based IRFs
- Choosing the order of the GMA (ie the number of Gaussians):
  $=>$ compare marginal densities of GMAs of increasing order N
- Dealing with non-stationary data
  - First-difference
  - Can allow for deterministic trend

Back to monetary policy

Are there non-linearities in the effects of monetary shocks?

- Small-scale model similar to Primiceri (2005)
- (U, Π, r) over 1959-2007
- Unemployment rate, PCE price index, fed funds rate

Asymmetry: IRFs from GMA(2)



Robustness

Same results when:

- Using output growth
- Using detrended output

Asymmetry and state-dependence

- GMA(1)

$$\psi^+(k) = (1 + \gamma^+ z_{t-k}) a^+ e^{-\left(\frac{k-b^+}{c^+}\right)^2}$$

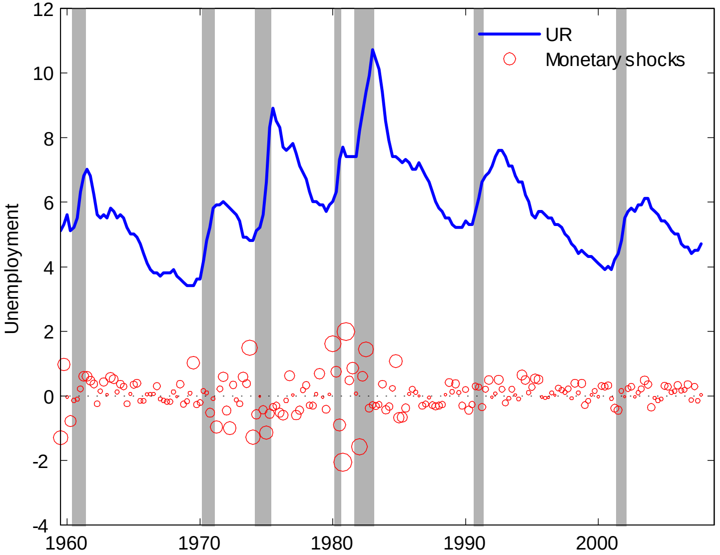
- Similarly for $\psi^-(k)$
- z_t: Unemployment rate

Marginal data densities

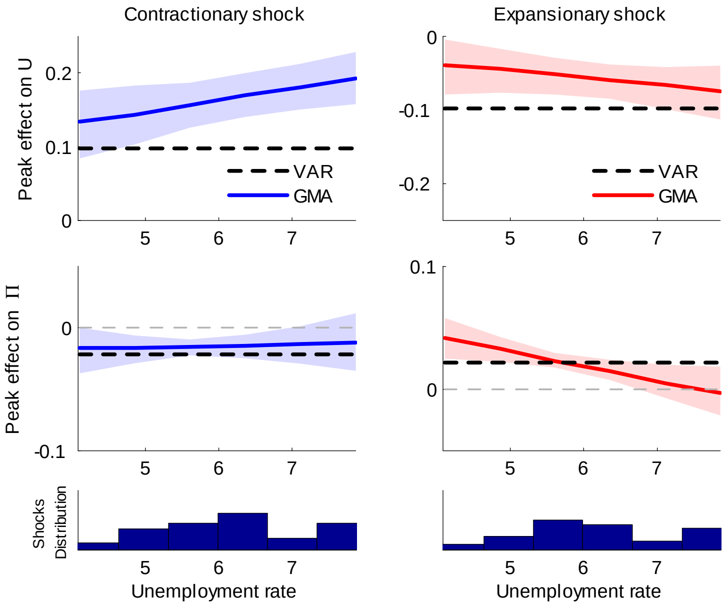
	VAR	GMA(1)	GMA(1) Asymmetry	GMA(2) Asymmetry	GMA(3) Asymmetry	GMA(1) Asymmetry State dep.
	(1)	**(2)**	**(3)**	**(4)**	**(5)**	**(6)**
(log) marginal data density	112	118	127	138	107	**158**

Note: Trivariate models with unemployment, PCE inflation and the fed funds rate estimated over 1959-2007. The VAR estimates and confidence bands are obtained from a Bayesian VAR with Normal-Whishart priors.

Distribution of monetary shocks over cycle



Economic slack and effects of monetary shocks



Robustness

Same results with alternative business cycle indicator z_t:

- Detrended unemployment
- Output growth

Take away: monetary policy has highly non-linear effects

- Asymmetry:
  - large effect of contractionary shocks
  - small effect of expansionary shocks
- State dependence:
  - In tight labor market, expansionary shock only generates inflation

Conclusion

- New method to identify the non-linear effects of shocks
- strong non-linearities in the effects of monetary policy shocks
- Can apply method to non-linear effects fiscal policy, credit supply shocks, etc..
- Not limited to just-identified models, can generalize to under-identified models (sign-restrictions)

Theories behind Non-linearities

- Asymmetry
  - Credit constraints
    $\longrightarrow$ Binding vs. non-binding
  - Downward wage (price) rigidity
- State dependence
  - Less inflationary pressure in periods of slack
    -> CB has more room to stimulate Y before P increases

Literature

1. Get non-linear effects from single equation regression (distributed lags model or Jorda's Local Projection)

   Asymmetry: Cover (1992), DeLong and Summers (1988), Morgan (1993).

   State-dependence: Thwaites and Tenreyro (2013), Santoro et al. (2014)

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2. Regime-switching models: allows the economy to respond differently depending on its state

   Beaudry and Koop (1993), Thoma (1994), Potter (1995), Kandil (1995), Koop, Pesaran and Potter (1996), Koop and Potter, (1998), Ravn and Sola (1996, 2004), Weise (1999), Lo and Piger (2002)

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   - Problem: more efficient but restrictive and not flexible
     -> switching variable is not the current shock, but a function of past shocks
     -> only a small number of states

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3. Angrist et al. (2013) semi-parametric method based on propensity score weighting

Approximating IRFs with mixtures of Gaussians (2)

Theorem

Let f be a continuous function over $[0,\infty[$ that satisfies $\int_0^{\infty} f(x)^2 dx < \infty$. Then, there exists a function f_N defined by

$$f_N(x) = \sum_{n=1}^{N} a_n e^{-\left(\frac{x-b_n}{c_n}\right)^2}$$

with a_n, b_n, $c_n \in \mathbb{R}$ for $n \in \mathbb{N}$, such that $f_N \longrightarrow f$ on every interval of $\mathbb{R}$.

Monte Carlo simulations

Illustrate performance of GMA in finite sample with 3 simulations:

1. Linear
2. Asymmetry
3. Asymmetry and state dependence

Lessons from Monte-Carlo simulations

- GMA performs as well as VAR in linear models
- Successfully detects asymmetry or state-dependence
- Delivers good estimates of magnitude of non-linearities

Simulations

- - Estimate a VAR model
  - Invert VAR to get $\hat{\Psi}_k$
  - Modify $\hat{\Psi}_k$ to introduce non-linearity (asym, state dep)
  - Simulate data using Normal shocks

Simulations

- - Estimate a VAR model
  - Invert VAR to get $\hat{\Psi}_k$
  - Modify $\hat{\Psi}_k$ to introduce non-linearity (asym, state dep)
  - Simulate data using Normal shocks
- In all sims, GMA is misspecified:
  - Conservative
  - Consistent with "GMA meant to *approximate* true DGP, and yet can recover non-linearities"

DGP #1: Linear case

- Use trivariate VAR with $(\mathrm{U}, \Pi, \mathrm{r})$ over 1959-2007
- Simulate data with $T = 200$ over 50 replications
- Compare MSE, avg length and coverage rate of GMA vs. VAR

Summary statistics of simulation #1

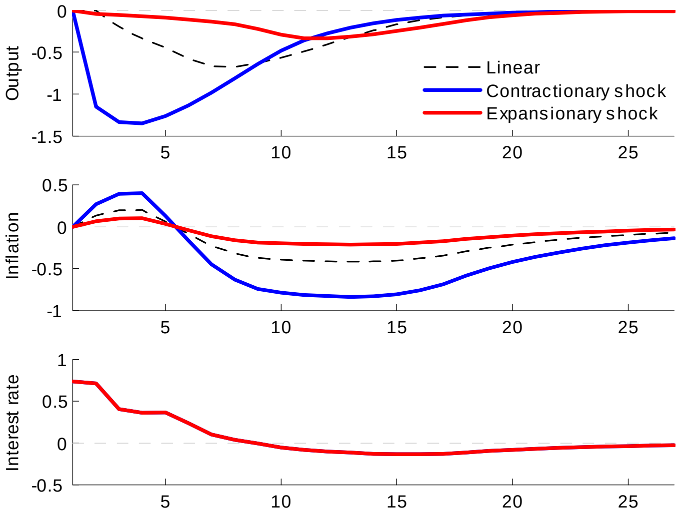
Table 1: Summary statistics for Monte Carlo simulation with linear model

	U				FFR	
	VAR	**GMA**	**VAR**	**GMA**	**VAR**	**GMA**
MSE	**0.057**	**0.043**	**0.077**	**0.041**	**0.003**	**0.002**
Avg length (at peak effect)	0.16	0.13	0.27	0.11	0.05	0.03
Coverage rate (at peak effect)	0.94	0.83	1	0.78	0.94	0.93

Note: Summary statistics over 50 Monte-Carlo replications. MSE is the mean-squared error of the estimated impulse response function over horizons 1 to 25. Avg length is the average distance between the lower (2.5%) and upper (97.5%) confidence bands at the time of peak effect of the monetary shock. The coverage rate is the frequency with which the true value lays within 95 percent of the posterior distribution. The VAR estimates and confidence bands are obtained from a Bayesian VAR with Normal-Whishart priors. U, Π and FFR denote respectively unemployment, inflation and the fed funds rate.

DGP #2: Asymmetry

(lnY,dlnP,r) in response to monetary shock



Summary statistics of simulation #2

Table 2: Summary statistics for Monte Carlo exercise with asymmetric model

	a^+-a^-		
	y	**dp**	**ffr**
Mean **(true value)**	**-0.82** **(-1.00)**	**-0.50** **(-0.60)**	**0.03** **(0.00)**
Std-dev	0.28	0.17	0.12
Frequency of rejection of zero coefficient	**0.94**	0.90	0.08
Coverage rate	**0.82**	0.86	0.88

Note: Summary statistics over 50 Monte-Carlo replications. For each coefficient of interest, "Frequency of rejection of zero coefficient" is the frequency that 0 lies outside 90 percent of the posterior distribution, and "Coverage rate" is the frequency with which the true value lies within 90 percent of the posterior distribution. y, **dp** and ffr denote respectively output, inflation and the fed funds rate.

DGP #3: Asymmetry and state dep.

- idem as DGP #1 but in addition

$$\gamma_y^+ > 0$$

  with z_t the US unemployment rate

- A positive monetary shock has a larger effect on output in recessions

Summary statistics of simulation #3

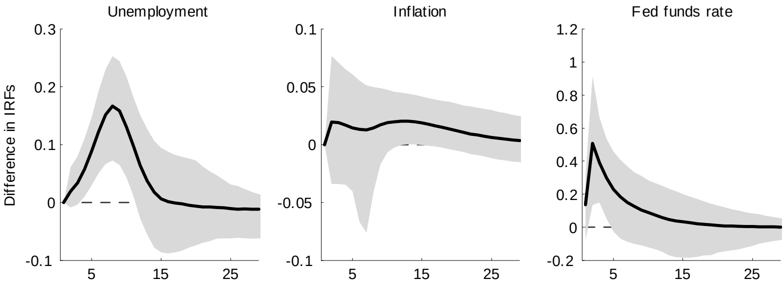
Table 3: Summary statistics for Monte Carlo exercise with asymmetry and state dependence

	γ^{+}-γ^{-}		γ^{+}		γ^{-}		α^{+}-α^{-}	
	y	**dp**	**y**	**dp**	**y**	**dp**	**y**	**dp**
Mean (true value)	**0.96** **(1.00)**	**0.02** **(0.00)**	0.71 (1.00)	0.00 (0.00)	-0.21 (0.00)	-0.00 (0.00)	**-0.78** **(-1.00)**	**-0.48** **(-0.60)**
Std-dev	0.26	0.17	0.31	0.19	0.23	0.19	0.37	0.23
Frequency of rejection of zero coefficient	**0.96**	**0.03**	0.87	0.06	0.20	0.05	**0.82**	**0.80**
Coverage rate	0.84	0.92	0.68	0.92	0.65	0.90	0.71	0.70

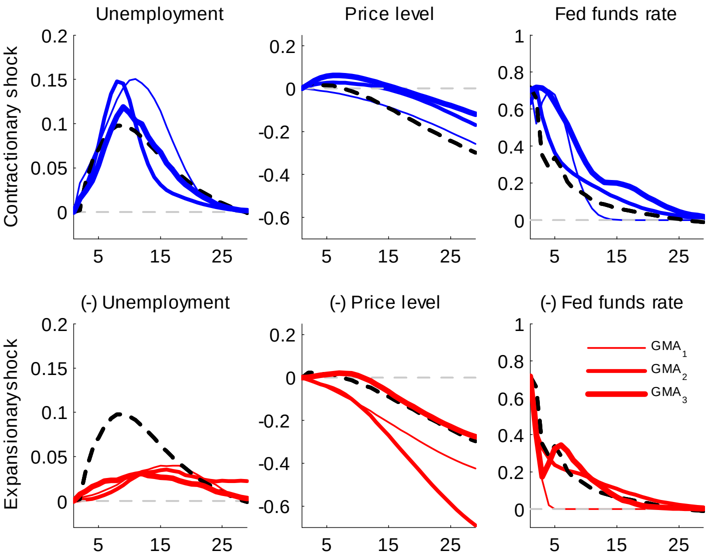
Note: Summary statistics over 50 Monte-Carlo replications. For each coefficient of interest, "Frequency of rejection of zero coefficient" is the frequency that 0 lies outside 90 percent of the posterior distribution, and "Coverage rate" is the frequency with which the true value lies within 90 percent of the posterior distribution. y and dp denote respectively output, inflation and the fed funds rate.

Testing for asymmetry

Posterior distribution of difference between IRFs to positive and negative shocks



GMA(1) to GMA(3)

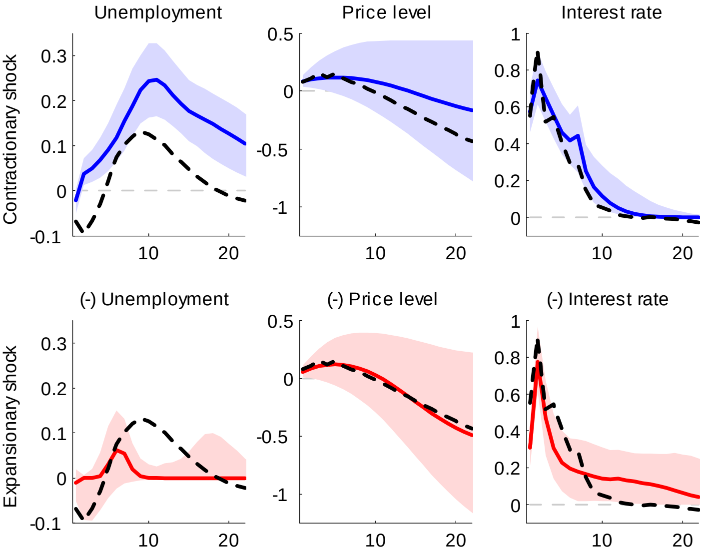


Unemployment
Price level
Fed funds rate
Contractionary shock
0.2
0.15
0.1
0.05
0
5
15
25
0.2
0
-0.2
-0.4
-0.6
1
0.8
0.6
0.4
0.2
(-) Unemployment
(-) Price level
(-) Fed funds rate
Expansionaryshock
GMA1
GMA2
GMA3

Using Romer and Romer shocks

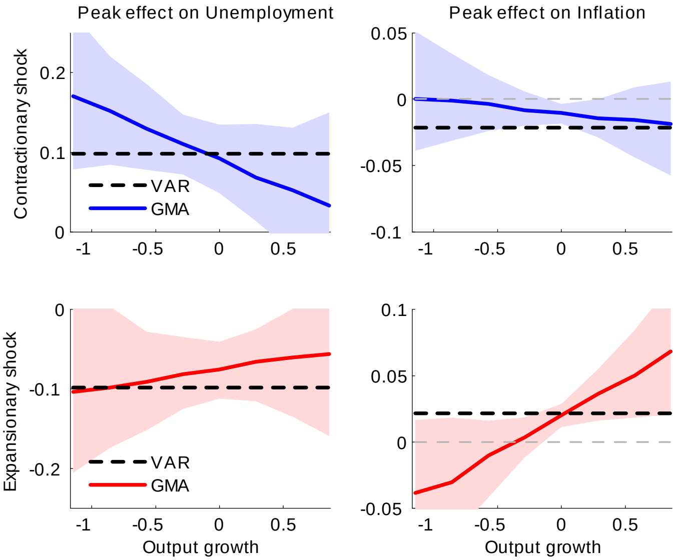
- Using Romer and Romer (2004) shocks ε^{MP} over 1976-2007
- Consider $\mathbf{Y} = (\varepsilon^{MP}, U, \Pi, r)$ with recursive ordering

Using Romer and Romer shocks



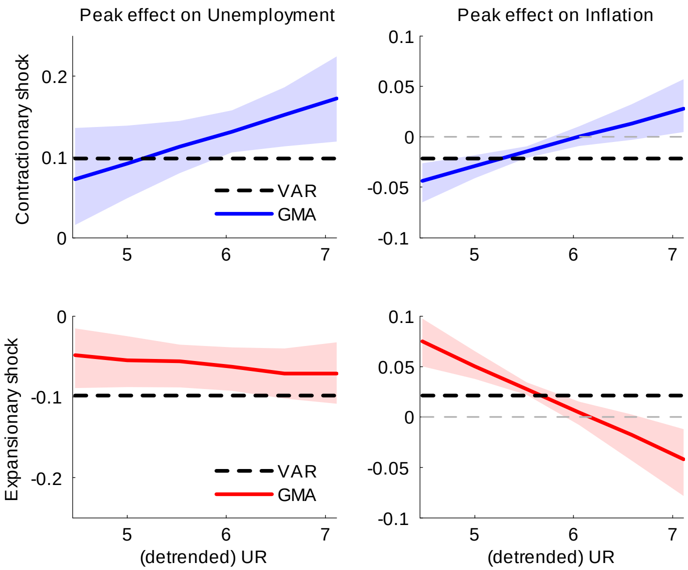
Peak effect and state of business cycle

Output growth



Peak effect and state of business cycle

detrended UR



Peak effect on FFR and state of business cycle