Optimal Forecasts in the Presence of Structural Breaks

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Two approaches to modelling parameter instability:
(1) Continuous break process and (2) discrete break process.
Under **discrete break** process a forecaster has a range of options:

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- Estimate the break date and size; use an optimal estimation window (Pesaran and Timmermann 2007, JoE)
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  - No estimates of the break date and size are necessary; the method works for any model
  - Good information about the break date and size can lead to better forecasts
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- Exponential smoothing forecasts are highly sensitive to the down-weighting parameter.
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For more regressors, the values of the weights depend on the regressors.
We also extend the optimal weights to situations where the break date and size are not known.
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We provide Monte Carlo to assess the influence of uncertainty around the break date and size.

Finally, we provide an application to forecasting real GDP across 9 advanced and emerging economies using the yield curve.
Consider the linear regression model

$$ y_t = \beta_t' x_t + \sigma_t \varepsilon_t, \quad \varepsilon_t \sim iid(0, 1), \quad t = 1, 2, \ldots, T, T + 1 \quad (1) $$

$\beta_t$ and $\sigma_t^2$ are subject to breaks.

The breaks can be continuous: $\beta_t$ changes its value in every period. For example:

$$ \beta_t = \beta_{t-1} + S_\beta v_t, \quad \text{where } v_t \sim iid(0, I_k), $$
Alternatively, the breaks could be **discrete**: parameters change at distinct points in time, \( T_{b,i}, i = 1, 2, \ldots, n, \)

\[
\beta_t = \begin{cases} 
\beta_{(1)} & \text{for } 1 < t \leq T_{b,1} \\
\beta_{(2)} & \text{for } T_{b,1} < t \leq T_{b,2} \\
& \vdots \\
\beta_{(n)} & \text{for } T_{b,n} < t \leq T
\end{cases}
\]

Additionally, \( \sigma_t \) may be subject to a similar break process. The number of discrete breaks, \( n \), is assumed to be small. The break sizes, \( \| \beta_{(i)} - \beta_{(i-1)} \| \), could be large relative to \( \sigma_t \).
The choice between specifications depends on the particular forecasting problem under consideration.
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We propose a general approach to achieve a minimum mean square forecast error: Weigh past observations by weights $w_t$ in the estimation

$$
\hat{\beta}_T(w) = \left( \sum_{t=1}^{T} w_t x_t x_t' \right)^{-1} \sum_{t=1}^{T} w_t x_t y_t,
$$

subject to $\sum_{t=1}^{T} w_t = 1$.  

**Pesaran, Pick and Pranovich**

Optimal forecasts in the presence of structural breaks
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subject to $\sum_{t=1}^{T} w_t = 1$.

The weights $w = (w_1, w_2, \ldots, w_T)'$ are chosen such that the MSFE of the one-step ahead forecast $\hat{y}_{T+1} = \hat{\beta}_T' x_{T+1}$ is minimized.
Consider the model

\[ y_t = \beta_t + \sigma \varepsilon_t, \]  

(2)

and assume that \( \beta_t \) is subject to a single, discrete break at \( T_b \), \( 1 < T_b < T \),

\[ \beta_t = \begin{cases} 
\beta_{(1)} & \text{for } t \leq T_b \\
\beta_{(2)} & \text{for } T_b < t \leq T + 1 
\end{cases} \]
Optimal weights in a model with a single, discrete break

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\beta_2 & \text{for } T_b < t \leq T + 1
\end{cases} \]

In this case the forecast is \( \hat{y}_{T+1} = \hat{\beta}_T(w) \) where

\[ \hat{\beta}_T(w) = \sum_{t=1}^{T} w_t y_t \]

and

\[ \hat{\beta}_T(w) - \beta_T = (\beta_1 - \beta_2) \sum_{t=1}^{T_b} w_t + \sum_{t=1}^{T} w_t \sigma \varepsilon_t. \]
The MSFE scaled by the error variance is

\[ E[\sigma_{\varepsilon}^{-2}e_{T+1}^2(w)] = 1 + \lambda^2 \left( \sum_{t=1}^{T_b} w_t \right)^2 + \sum_{t=1}^{T} w_t^2, \]  

(3)

where \( \lambda = (\beta_{(1)} - \beta_{(2)})/\sigma_{\varepsilon} \).
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(3)

where $\lambda = (\beta_{(1)} - \beta_{(2)}) / \sigma_e$.

The optimal weights, obtained by minimizing (3) subject to $\sum_{t=1}^{T} w_t = 1$, yields

$$w^{(1)} = \frac{1}{T} \frac{1}{1 + Tb(1 - b)\lambda^2},$$

(4)

and

$$w^{(2)} = \frac{1}{T} \frac{1 + Tb\lambda^2}{1 + Tb(1 - b)\lambda^2}.$$

(5)

where $b = T_b / T$. 
Comparison to alternative forecasts

- Using post-break observations.
- Optimal estimation window (Pesaran and Timmermann 2007, JoE): Use window that results in minimum MSFE.
- Averaging over estimation windows (Pesaran and Pick 2011, JBES): Average over potential optimal windows to obtain robust forecast.
- Exponential smoothing (Holt 1957): Down-weighting for discrete break processes.
### Exact relative MSFE for a single discrete, break for known $b$ and $\lambda$

<table>
<thead>
<tr>
<th></th>
<th>0.5</th>
<th>1</th>
<th>2</th>
<th>0.5</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T = 100$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$b$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$0.95$</td>
<td>0.901</td>
<td>0.610</td>
<td>0.258</td>
<td>0.884</td>
<td>0.600</td>
<td>0.258</td>
</tr>
<tr>
<td>$\lambda$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.5</td>
<td>0.971</td>
<td>0.628</td>
<td>0.260</td>
<td>0.907</td>
<td>0.604</td>
<td>0.259</td>
</tr>
<tr>
<td>1</td>
<td>0.939</td>
<td>0.622</td>
<td>0.259</td>
<td>0.899</td>
<td>0.603</td>
<td>0.259</td>
</tr>
<tr>
<td>2</td>
<td>0.966</td>
<td>0.900</td>
<td>0.829</td>
<td>0.941</td>
<td>0.830</td>
<td>0.704</td>
</tr>
<tr>
<td>AveW($\nu_{\text{min}} = 0.05$)</td>
<td>0.973</td>
<td>0.924</td>
<td>0.872</td>
<td>0.958</td>
<td>0.883</td>
<td>0.799</td>
</tr>
<tr>
<td>ExpS($\gamma = 0.95$)</td>
<td></td>
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</tbody>
</table>
Consider the linear regression model

\[ y_t = \beta_t' x_t + \sigma \varepsilon_t \]

where the parameter vector \( \beta_t \) is subject to \( n \) breaks at break points \( b_i = \frac{T_{b,i}}{T} \), such that \( b_1 < b_2 < \ldots < b_n \).

Again, initially assume that \( n = 2 \), such that the parameter vector is

\[
\beta_t = \begin{cases} 
\beta_{(1)} & \text{for } 1 < t \leq T_{b,1} \\
\beta_{(2)} & \text{for } T_{b,1} < t \leq T_{b,2} \\
\beta_{(3)} & \text{for } T_{b,2} < t \leq T 
\end{cases}
\]
Patterns of weights across regimes

- The pattern of the optimal weights depends on the pattern of the slope coefficients across the regimes.

If the slopes are rising or falling monotonically (\(1 > 2 > 3\) or \(1 < 2 < 3\)) the optimal weights also decay monotonically (\(w(1) < w(2) < w(3)\)), which is in line with down-weighting of observations. When this is not the case, it is possible for the middle regime to get less weight than the first and the third regimes. Regimes with the same slope coefficients get the same weights.
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- When this is not the case, it is possible for the middle regime to get less weight than the first and the third regimes.
- Regimes with the same slope coefficients get the same weights.
Figure: Regime-specific slopes: $\beta_1 = 3$, $\beta_2 = 1$, and $\beta_3 = 2.5$
Optimal weights when the time and size of the break are uncertain

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- In particular, the size of the break is difficult to estimate unless a considerable number of post-break observations is available.
- Need to obtain weights that allow for breaks but are robust to the time and size of break(s).
For large values of $T$ we get the following approximation

$$w(a, b, q^2, \phi^2) = \frac{1}{T} \left[ \frac{1}{1 - b} - \frac{1}{1 - b} I(b - a) \right]$$

$$- \frac{1}{T^2 \phi^2} \frac{1}{(1 - b)^2} + \frac{1}{T^2 \phi^2} \frac{1}{b(1 - b)^2} I(b - a) + O(T^{-3}).$$

where $\phi^2 = \lambda^2 \hat{\omega}_{x,1}^2$ and $q^2 = \sigma_{(1)}^2 / \sigma_{(2)}^2$. 
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- The first order term does not depend on the break size or the ratio of error variances.
- The terms up to order $T^{-2}$ are independent of $q^2$: a break in the variance is dominated by a break in the mean and slope parameters.
For large $T$ robust weights can be obtained by integrating over $b \sim U(\underline{b}, \bar{b})$.

$$w(a) \approx \begin{cases} 0, & \text{if } a < \underline{b} \\ \frac{-1}{T(b-\bar{b})} \log \left( \frac{1-a}{1-b} \right), & \text{if } \underline{b} \leq a \leq \bar{b} \\ \frac{-1}{T(\bar{b}-\bar{b})} \log \left( \frac{1-\bar{b}}{1-b} \right), & \text{if } a > \bar{b} \end{cases}$$
Figure: Approximate optimal weights for break in variance, $T = 100, \underline{b} = 0.3, \bar{b} = 0.9$
If $b$ and $\bar{b}$ are close to 0 and 1, we have

$$w(a) \approx \frac{-\log(1 - a)}{T}, \quad a \in [0, \bar{b}].$$
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• A discrete time version is given by

$$w_t = \frac{-\log(1 - t/T)}{T}, \text{ for } t = 1, 2, \ldots, T - 1$$
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The value of $w_T$ is not defined, but can be computed using $t = T - 0.5$, namely

$$w_T = \frac{-\log(0.5/T)}{T}$$
Optimal weights in a model with a single, discrete break

- If $b$ and $\bar{b}$ are close to 0 and 1, we have
  \[ w(a) \approx \frac{-\log(1 - a)}{T}, \quad a \in [0, \bar{b}] \, . \]

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  \[ w_T = \frac{-\log(0.5/T)}{T} \]

- The resulting weights are
  \[ w_t^* = \frac{w_t}{\sum_{j=1}^{T} w_j}, \quad \text{for } t = 1, 2, \ldots, T \]
Consider the case of two breaks. Numerical solutions by integrating over a grid for $b_1$ and $b_2$.

The figure plots the robust weights for two breaks and $T = 100$, where the first graph reports the weights for $\phi_1 = -0.5$ and $\phi_2 = 1.5$, the second for $\phi_1 = 0$ and $\phi_2 = 1$, the third for $\phi_1 = 2$ and $\phi_2 = 1$. 

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Optimal forecasts in the presence of structural breaks
In practice, given that the break date is uncertain, the size of break is also likely to be unknown. In addition to the break date, we therefore also integrate over the break sizes in the weights. The figure plots the weights when $\phi_1$ and $\phi_2$ are integrated with respect to a uniform distribution in the range $-2$ to $2$. The first graph shows the weights for $T = 50$ and the second for $T = 200$. 
Monte Carlo experiments

The first model considered is

\[ y_t = \mu_t + \sigma_t \varepsilon_t, \quad \varepsilon_t \sim N(0, 1) \quad t = 1, 2, \ldots, T, T + 1 \]

\[ \mu_t = \mu_{t-1} + \sigma_v \nu_t, \quad \nu_t \sim N(0, 1) \]

\[ T = 50, 100, 200, \text{ and } \gamma = \{0.8, 0.9, 0.95, 0.98\}, \text{ which corresponds to } \delta = \sigma_\varepsilon / \sigma_v \approx \{4.471, 9.487, 19.494, 49.497\}. \]
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Next, we assume that the mean has a discrete break

\[ \mu_t = \begin{cases} 
\mu_{(1)} & t \leq T_b \\
\mu_{(2)} & t > T_b 
\end{cases} \quad \text{and} \quad \sigma_t = \begin{cases} 
\sigma_{(1)} & t \leq T_b \\
\sigma_{(2)} & t > T_b 
\end{cases} \]

We set \( b = \{0.95, 0.9\} \), \( \lambda = (\mu_{(1)} - \mu_{(2)})/\sigma_{(2)} = \{0.5, 1, 2\} \) and \( q = \sigma_{(1)}/\sigma_{(2)} = \{0.5, 1\} \). We assume that \( T_b, \lambda \) and \( q \) are unknown and have to be estimated.
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We report the relative MSFE:

\[ \text{Re MSFE} = \frac{\text{MSFE}_i}{\text{MSFE}_{\text{equal } w}}, \quad i = \text{OptW, PostB, RobustW, } \ldots \]
Relative MSFE: Monte Carlo results for continuous breaks

\( T = 100 \) and \( q = 1 \)

<table>
<thead>
<tr>
<th>( \gamma )</th>
<th>0.8</th>
<th>0.9</th>
<th>0.95</th>
<th>0.98</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \delta )</td>
<td>4.472</td>
<td>9.487</td>
<td>19.494</td>
<td>49.497</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Method</th>
<th>( \gamma = 0.8 )</th>
<th>( \gamma = 0.9 )</th>
<th>( \gamma = 0.95 )</th>
<th>( \gamma = 0.98 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>opt.weight(cont.break; ( \delta ))</td>
<td>0.444</td>
<td>0.772</td>
<td>0.956</td>
<td>0.999</td>
</tr>
<tr>
<td>est.opt.weight(cont.break; ( \delta ))</td>
<td>0.455</td>
<td>0.794</td>
<td>0.995</td>
<td>1.022</td>
</tr>
<tr>
<td>ExpS(( \hat{\gamma} ))</td>
<td>0.455</td>
<td>0.794</td>
<td>0.995</td>
<td>1.022</td>
</tr>
<tr>
<td>ExpS(( \gamma = 0.95 ))</td>
<td>0.557</td>
<td>0.799</td>
<td>0.956</td>
<td>1.015</td>
</tr>
<tr>
<td>ExpS(( \gamma = 0.98 ))</td>
<td>0.744</td>
<td>0.885</td>
<td>0.968</td>
<td>1.000</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Method</th>
<th>( \gamma = 0.8 )</th>
<th>( \gamma = 0.9 )</th>
<th>( \gamma = 0.95 )</th>
<th>( \gamma = 0.98 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>est.opt.weight(disc.break; ( \hat{b}, \hat{\lambda} ))</td>
<td>0.510</td>
<td>0.856</td>
<td>1.085</td>
<td>1.121</td>
</tr>
<tr>
<td>rob.weights(( b = 0.75, \bar{b} = 0.98 ))</td>
<td>0.508</td>
<td>0.781</td>
<td>0.963</td>
<td>1.029</td>
</tr>
<tr>
<td>rob.weights(( b = 0, \bar{b} = 1 ))</td>
<td>0.620</td>
<td>0.829</td>
<td>0.958</td>
<td>1.007</td>
</tr>
<tr>
<td>rob.weights(two breaks)</td>
<td>0.761</td>
<td>0.888</td>
<td>0.969</td>
<td>1.000</td>
</tr>
<tr>
<td>post-break obs.(( \hat{b} ))</td>
<td>0.511</td>
<td>0.864</td>
<td>1.105</td>
<td>1.144</td>
</tr>
<tr>
<td>opt.window(( \hat{b}, \hat{\lambda} ))</td>
<td>0.503</td>
<td>0.828</td>
<td>1.042</td>
<td>1.081</td>
</tr>
<tr>
<td>AveW(( w_{\text{min}} = 0.05 ))</td>
<td>0.644</td>
<td>0.840</td>
<td>0.959</td>
<td>1.005</td>
</tr>
</tbody>
</table>
Relative MSFE: Monte Carlo results for a discrete break in drift

\( T = 100 \) and \( q = 1 \)

<table>
<thead>
<tr>
<th>( b )</th>
<th>( \lambda )</th>
<th>0.5</th>
<th>1</th>
<th>2</th>
<th>0.5</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>opt.weight(disc.break; ( b, \lambda ))</td>
<td>0.893</td>
<td>0.603</td>
<td>0.256</td>
<td>0.875</td>
<td>0.592</td>
<td>0.257</td>
<td></td>
</tr>
<tr>
<td>est.opt.weight(disc.break; ( \hat{b}, \hat{\lambda} ))</td>
<td>1.022</td>
<td>0.826</td>
<td>0.320</td>
<td>1.014</td>
<td>0.737</td>
<td>0.263</td>
<td></td>
</tr>
<tr>
<td>rob.weights(( b = 0.75, \bar{b} = 0.98 ))</td>
<td>0.934</td>
<td>0.796</td>
<td>0.648</td>
<td>0.901</td>
<td>0.705</td>
<td>0.480</td>
<td></td>
</tr>
<tr>
<td>rob.weights(( b = 0, \bar{b} = 1 ))</td>
<td>0.953</td>
<td>0.867</td>
<td>0.775</td>
<td>0.931</td>
<td>0.805</td>
<td>0.662</td>
<td></td>
</tr>
<tr>
<td>post-break obs.(( \hat{b} ))</td>
<td>1.039</td>
<td>0.839</td>
<td>0.319</td>
<td>1.030</td>
<td>0.747</td>
<td>0.262</td>
<td></td>
</tr>
<tr>
<td>opt.window(( \hat{b}, \hat{\lambda} ))</td>
<td>0.991</td>
<td>0.800</td>
<td>0.329</td>
<td>0.986</td>
<td>0.722</td>
<td>0.268</td>
<td></td>
</tr>
<tr>
<td>AveW(( w_{\text{min}} = 0.05 ))</td>
<td>0.965</td>
<td>0.900</td>
<td>0.830</td>
<td>0.940</td>
<td>0.831</td>
<td>0.706</td>
<td></td>
</tr>
<tr>
<td>est.opt.weight(cont.break; ( \hat{\delta} ))</td>
<td>0.992</td>
<td>0.944</td>
<td>0.666</td>
<td>0.984</td>
<td>0.847</td>
<td>0.337</td>
<td></td>
</tr>
<tr>
<td>ExpS(( \hat{\gamma} ))</td>
<td>0.992</td>
<td>0.944</td>
<td>0.666</td>
<td>0.984</td>
<td>0.847</td>
<td>0.337</td>
<td></td>
</tr>
<tr>
<td>ExpS(( \gamma = 0.95 ))</td>
<td>0.949</td>
<td>0.849</td>
<td>0.741</td>
<td>0.916</td>
<td>0.759</td>
<td>0.579</td>
<td></td>
</tr>
<tr>
<td>ExpS(( \gamma = 0.98 ))</td>
<td>0.980</td>
<td>0.944</td>
<td>0.905</td>
<td>0.963</td>
<td>0.899</td>
<td>0.826</td>
<td></td>
</tr>
</tbody>
</table>
For small breaks ($\lambda = 0.5$) the robust (optimal) weights that assume the break date to fall within a range perform best. Also both AveW and ExpS perform better than methods based on point estimates.

For larger breaks, the methods using point estimates improve over other methods.

For $\lambda = 1$ the optimal window forecast performs best, followed by the optimal weights forecast.

For $\lambda = 2$ the post break forecast performs best, closely followed by the optimal weights forecast.
The yield curve as a predictor of real economic activity

- The slope of the yield curve has emerged as a valuable leading indicator of GDP growth (Stock and Watson 2003).
- Recent evidence suggests that the relationship between GDP growth and the yield curve changes over time (Estrella, Rodrigues and Schich 2003, Giacomini and Rossi 2006, Schrimpf and Wang 2010).
- The base line specification of our forecast regression is

\[ y_{t,t+h} = \beta_0 + \beta_1 s_t + \varepsilon_t \]

where

\[ y_{t,t+h} = 100 \ln(Y_{t+h}/Y_t), \]

\( Y_t \) is the level of real GDP, the slope of the yield curve, \( s_t = i^L_t - i^S_t \), is the difference between the long term interest rate, \( i^L_t \), and the short term interest rate, \( i^S_t \).
We evaluate the forecasts for various horizons, $h = 1, 2, 3, 4$.

We use data on GDP and long and short term interest rates from the data set available with the GVAR toolbox (Smith and Galesi 2010).

We use data for 9 countries with long time series: Australia, Canada, France, Germany, Italy, Japan, Spain, UK, and USA.

The data are quarterly, start in 1979Q1, and end in 2009Q4.

Recursive out-of-sample forecasts are constructed with the first forecast for 1994Q1.
Predictive power of the yield curve: Relative forecast accuracy averaged across countries

<table>
<thead>
<tr>
<th></th>
<th>Equally weighted ave.</th>
</tr>
</thead>
<tbody>
<tr>
<td>h</td>
<td>1</td>
</tr>
<tr>
<td>1994Q1–2009Q4 (Full forecast evaluations sample)</td>
<td></td>
</tr>
<tr>
<td>equal weight (MSFE)</td>
<td>0.521</td>
</tr>
<tr>
<td>estim.opt.weight</td>
<td>1.056</td>
</tr>
<tr>
<td>rob.weight (1 break)</td>
<td>0.915</td>
</tr>
<tr>
<td>rob.weight (2 breaks)</td>
<td>0.953</td>
</tr>
<tr>
<td>post break</td>
<td>1.171</td>
</tr>
<tr>
<td>AveW</td>
<td>0.991</td>
</tr>
<tr>
<td>ExpS ($\gamma = 0.95$)</td>
<td>0.909</td>
</tr>
<tr>
<td>ExpS ($\gamma = 0.98$)</td>
<td>0.956</td>
</tr>
</tbody>
</table>
Conclusion

- Optimal weights for continuous and discrete break processes.
- Continuous break process: exponential smoothing.
- Discrete break process: new results.
- In practice, dates and sizes of breaks are unknown and estimates are highly unreliable: Robust weights
- Advantage of robust weights: Don’t need choice of down-weighting factor.
- Possible to improve robust weights using some information on breaks, such as time interval over which breaks are likely.