The new Keynesian model with imperfect information and learning

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Abstract

The standard version of the new Keynesian (NK) model has important, well known, empirical limitations, in particular with regard to inflation and output dynamics following a monetary shock. Recent extensions fare better empirically but only because they incorporate some controversial features. We demonstrate that the NK model can be improved without sacrificing full rationality. With the introduction of a modest amount of imperfect information and gradual learning the model can generate inflation persistence, realistic inflation and output dynamics and a liquidity effect. Moreover, this feature improves the overall fitness of the model –according to standard moments criteria– relative to the standard version.

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Introduction

The New Keynesian model has gained wide acceptance. Nevertheless, the model has a number of important implications that seem to be at variance with the empirical evidence.

Mankiw and Reis (2002) argue that it cannot produce plausible inflation and output dynamics following a monetary shock, in particular, the delayed, hump shaped response of inflation documented by Christiano, Eichenbaum and Evans (2005). That it is inconsistent with the accelerator hypothesis (the positive relation between economic activity and the change in the inflation rate). And that it cannot generate serial correlation in inflation forecast errors. Gali (2003) shows that the model cannot generate a liquidity effect following a monetary shock. And so on. Several modification have suggested in the literature with the aim of rectifying these important weaknesses.

One involves the full or partial abandonment of rational expectations. For instance, Gali and Gertler (1999) and Christiano, Eichenbaum and Evans (2005) assume non–optimal price indexation schemes for a subset of the population. In particular, they assume that a fraction of the population adjusts prices in a backward looking way[1]. Roberts (2001) and Ireland (2000) recommend the use of adaptive expectations. Such specifications generate a Phillips curve that contains lagged inflation. This gives rise to a delayed inflation response and inflation persistence.

Another approach, due to Mankiw and Reis (2002) maintains rational expectations but assumes that information is sticky. Information disseminates slowly throughout the population with the result that different agents' expectations are based on different information sets. The resulting Phillips curve contains past expectations of current economic conditions, which again gives rise to inertial inflation behavior.

A third one, introduces modifications on the real side of the model in order to generate sluggishness in real marginal costs. Christiano, Eichenbaum and Evans (2005) include various types of adjustment costs, namely, habit formation, adjustment costs on investment, and variable capital utilization. Even with these modifications, though, the new Keynesian model needs a backward looking price indexation scheme to produce satisfactory empirical performance (Collard and Dellas, 2005).

Finally, a forth approach, maintains rational expectations and the standard structure but emphasizes imperfect information about the shocks and learning (Dellas, 2004). This approach has already been fruitfully applied to the study of exceptional inflation episodes: The great inflation of the 70s (Bullard and Eusepi, 2003, Collard and Dellas, 2004) and the disinflation of the 80s

[1]Minford and Peel (2004) argue that allowing such agents to adjust prices based on expected rather than on past inflation eliminates the new Phillips curve.

(Erceg and Levin, 2003). These papers demonstrate how imperfect information can be a source of output and inflation persistence, because the serial correlation in inflation forecast errors adds a persistent, disturbance term to the Phillips. They do not address, however, the issue of the dynamic effects of monetary shocks on inflation[2]. As Mankiw and Reis (2002) remark, the key empirical failure of the new Keynesian model is not inflation persistence per se (which can be generated by persistence in money growth) but rather the delayed response of inflation to monetary policy shocks. More importantly, these papers focus exclusively on the ability of the model to address a particular question and, consequently, stay short of assessing its *overall* performance[3]. This is an important omission as often success at one front comes at the expense of success along some other dimension. It thus remains an open question whether imperfect information and learning can help the new Keynesian model not only better match some of the important stylized facts described above but also achieve more satisfactory overall performance (according, for instance, to standard validation criteria).

This paper addresses this issue. We add imperfect information and gradual learning to the prototype new Keynesian model. In particular, we assume that the agents observe some variables with error and can only gradually learn about the true values over time (based on the Kalman filter). We then evaluate the ability of the model to reproduce the stylized facts –mentioned above– that have proved too much of a challenge under perfect information. And also, its ability to match various moments.

We find that a specification with a modest amount of imperfect information is considerably superior to the standard, full information version. The model can produce a weak instantaneous response to current shocks. A delayed, hump shaped response of inflation and output following a monetary shock. Seriously correlated inflation forecast errors. And a liquidity effect following a monetary shock. Moreover, the model's ability to match the data in terms of various moments seems superior to that of the perfect information version.

The remaining of the paper is organized as follows. Section 1 presents the model. Section 2 discusses the calibration. Section 3 presents the main results. A last section concludes.

1 The model

The set up is the standard NNS model. The economy is populated by a large number of identical infinitely–lived households and consists of two sectors: one producing intermediate goods and the other a final good. The intermediate good is produced with capital and labor and the final

[2] Dellas, 2004, is an exception. Woodford, 2002, examines the dynamic effects of nominal shocks under gradual learning but in a model without price or wage staggering, a key feature of the new Keynesian model.

[3] The same lack of overall evaluation is also found in the sticky information papers mentioned above.

good with intermediate goods. The final good is homogeneous and can be used for consumption (private and public) and investment purposes.

1.1 The Household

Household preferences are characterized by the lifetime utility function:[4]

$$\sum_{\tau=0}^{\infty} E_t \beta^\tau U\left(C_{t+\tau}, \frac{M_{t+\tau}}{P_{t+\tau}}, \ell_{t+\tau}\right) \tag{1}$$

where $0 < \beta < 1$ is a constant discount factor, C denotes the domestic consumption bundle, M/P is real balances and ℓ is the quantity of leisure enjoyed by the representative household. The utility function,$U\left(C, \frac{M}{P}, \ell\right) : \mathbb{R}_+ \times \mathbb{R}_+ \times [0,1] \longrightarrow \mathbb{R}$ is increasing and concave in its arguments.

The household is subject to the following time constraint

$$\ell_t + h_t = 1 \tag{2}$$

where h denotes hours worked. The total time endowment is normalized to unity.

In each and every period, the representative household faces a budget constraint of the form

$$B_{t+1} + M_t + P_t(C_t + I_t + T_t) \quad \leq \quad R_{t-1}B_t + M_{t-1} + N_t + \Pi_t + P_t W_t h_t + P_t z_t K_t \tag{3}$$

where W_t is the real wage; P_t is the nominal price of the domestic final good;.C_t is consumption and I is investment expenditure; K_t is the amount of physical capital owned by the household and leased to the firms at the real rental rate z_t. M_{t-1}) is the amount of money that the household brings into period t, and M_t is the end of period t money holdings. N_t is a nominal lump–sum transfer received from the monetary authority; T_t is the lump–sum taxes paid to the government and used to finance government consumption.

Capital accumulates according to the law of motion

$$K_{t+1} = I_t - \frac{\varphi}{2}\left(\frac{I_t}{K_t} - \delta\right)^2 K_t + (1-\delta)K_t \tag{4}$$

where $\delta \in [0,1]$ denotes the rate of depreciation. The second term captures the existence of capital adjustment costs. $\varphi > 0$ is the capital adjustment costs parameter.

The household determines her consumption/savings, money holdings and leisure plans by maximizing her utility (1) subject to the time constraint (2), the budget constraint (3) and taking the evolution of physical capital (4) into account.

[4] $E_t(.)$ denotes mathematical conditional expectations. Expectations are conditional on information available at the beginning of period t.

1.2 Final sector

The final good is produced by combining intermediate goods. This process is described by the following CES function

$$Y_t = \left(\int_0^1 X_t(i)^\theta \mathrm{d}i \right)^{\frac{1}{\theta}} \tag{5}$$

where $\theta \in (-\infty, 1)$. θ determines the elasticity of substitution between the various inputs. The producers in this sector are assumed to behave competitively and to determine their demand for each good, $X_t(i)$, $i \in (0,1)$ by maximizing the static profit equation

$$\max_{\{X_t(i)\}_{i \in (0,1)}} P_t Y_t - \int_0^1 P_t(i) X_t(i) \mathrm{d}i \tag{6}$$

subject to (5), where $P_t(i)$ denotes the price of intermediate good i. This yields demand functions of the form:

$$X_t(i) = \left(\frac{P_t(i)}{P_t} \right)^{\frac{1}{\theta - 1}} Y_t \text{ for } i \in (0,1) \tag{7}$$

and the following general price index

$$P_t = \left(\int_0^1 P_t(i)^{\frac{\theta}{\theta-1}} \mathrm{d}i \right)^{\frac{\theta-1}{\theta}} \tag{8}$$

The final good may be used for consumption — private or public — and investment purposes.

1.3 Intermediate goods producers

Each firm i, $i \in (0,1)$, produces an intermediate good by means of capital and labor according to a constant returns–to–scale technology, represented by the Cobb–Douglas production function

$$X_t(i) = A_t K_t(i)^\alpha h_t(i)^{1-\alpha} \text{ with } \alpha \in (0,1) \tag{9}$$

where $K_t(i)$ and $h_t(i)$ respectively denote the physical capital and the labor input used by firm i in the production process. A_t is an exogenous stationary stochastic technology shock, whose properties will be defined later. Assuming that each firm i operates under perfect competition in the input markets, the firm determines its production plan so as to minimize its total cost

$$\min_{\{K_t(i), h_t(i)\}} P_t W_t h_t(i) + P_t z_t K_t(i)$$

subject to (9). This leads to the following expression for total costs:

$$P_t S_t X_t(i)$$

where the real marginal cost, S, is given by $\frac{W_t^{1-\alpha} z_t^\alpha}{\chi A_t}$ with $\chi = \alpha^\alpha (1-\alpha)^{1-\alpha}$

Intermediate goods producers are monopolistically competitive, and therefore set prices for the good they produce. We follow Calvo, 1983, in assuming that firms set their prices for a stochastic number of periods. In each and every period, a firm either gets the chance to adjust its price (an event occurring with probability γ) or it does not. In order to maintain long term money neutrality (in the absence of monetary frictions) we also assume that the price set by the firm grows at the steady state rate of inflation. Hence, if a firm i does not reset its price, the latter is given by $P_t(i) = \overline{\pi} P_{t-1}(i)$. A firm i sets its price, $\widetilde{p}_t(i)$, in period t in order to maximize its discounted profit flow:

$$\max_{\widetilde{p}_t(i)} \widetilde{\Pi}_t(i) + E_t \sum_{\tau=1}^{\infty} \Phi_{t+\tau}(1-\gamma)^{\tau-1} \left(\gamma \widetilde{\Pi}_{t+\tau}(i) + (1-\gamma)\Pi_{t+\tau}(i)\right)$$

subject to the total demand it faces

$$X_t(i) = \left(\frac{P_t(i)}{P_t}\right)^{\frac{1}{\theta-1}} Y_t$$

and where $\widetilde{\Pi}_{t+\tau}(i) = (\widetilde{p}_{t+\tau}(i) - P_{t+\tau}S_{t+\tau})X(i, s^{t+\tau})$ is the profit attained when the price is reset, while $\Pi_{t+\tau}(i) = (\overline{\pi}^{\tau}\widetilde{p}_t(i) - P_{t+\tau}S_{t+\tau})X_{t+\tau}(i)$ is the profit attained when the price is maintained. $\Phi_{t+\tau}$ is an appropriate discount factor related to the way the household values future as opposed to current consumption. This leads to the price setting equation

$$\widetilde{p}_t(i) = \frac{1}{\theta} \frac{E_t \sum_{\tau=0}^{\infty} \left[(1-\gamma)\overline{\pi}^{\frac{1}{\theta-1}}\right]^{\tau} \Phi_{t+\tau} P_{t+\tau}^{\frac{2-\theta}{1-\theta}} S_{t+\tau} Y_{t+\tau}}{E_t \sum_{\tau=0}^{\infty} \left[(1-\gamma)\overline{\pi}^{\frac{\theta}{\theta-1}}\right]^{\tau} \Phi_{t+\tau} P_{t+\tau}^{\frac{1}{\theta-1}} Y_{t+\tau}} \tag{10}$$

Since the price setting scheme is independent of any firm specific characteristic, all firms that reset their prices will choose the same price.

In each period, a fraction γ of contracts ends, so there are $\gamma(1-\gamma)$ contracts surviving from period $t-1$, and therefore $\gamma(1-\gamma)^j$ from period $t-j$. Hence, from (8), the aggregate intermediate price index is given by

$$P_t = \left(\sum_{i=0}^{\infty} \gamma(1-\gamma)^i \left(\frac{\widetilde{p}_{t-i}}{\overline{\pi}^i}\right)^{\frac{\theta}{\theta-1}}\right)^{\frac{\theta-1}{\theta}} \tag{11}$$

1.4 The monetary authorities

We use two alternative specifications of monetary policy: *(i)* an exogenous money supply rule and *(ii)* a standard Henderson–McKibbin–Taylor (HMT) rule. Under the former, the money supply is assumed to evolve according to

$$M_t = \exp(\mu_t) M_{t-1} \tag{12}$$

where the gross growth rate of the money supply, μ_t, is assumed to follow an exogenous stochastic process whose properties will be defined later.

Under the latter, the growth of the money supply is selected in order to satisfy

$$\widehat{R}_t = \rho_r \widehat{R}_{t-1} + (1-\rho_r)[k_\pi E_t(\widehat{\pi}_{t+1} - \pi) + k_y(\widehat{y}_t - y_t^\star)] \tag{13}$$

where $\widehat{\pi}_t$ and $\widehat{y}_t$ are actual output and expected gross inflation in logs respectively and π and $y_t^\star$ are the inflation and output targets respectively. The output target is set equal to potential output and the inflation target to the steady state rate of inflation. Potential output is not observable and the monetary authorities must learn about it.

1.5 The government

The government finances government expenditure on the domestic final good using lump sum taxes. The stationary component of government expenditures is assumed to follow an exogenous stochastic process, whose properties will be defined later.

1.6 The equilibrium

We now turn to the description of the equilibrium of the economy.

Definition 1 *An equilibrium of this economy is a sequence of prices* $\{\mathcal{P}_t\}_{t=0}^\infty = \{W_t, z_t, P_t, R_t, P_t(i), i \in (0,1)\}_{t=0}^\infty$ *and a sequence of quantities* $\{\mathcal{Q}_t\}_{t=0}^\infty = \{\{\mathcal{Q}_t^H\}_{t=0}^\infty, \{\mathcal{Q}_t^F\}_{t=0}^\infty\}$ *with*

$$\begin{aligned}\{\mathcal{Q}_t^H\}_{t=0}^\infty &= \{C_t, I_t, B_t, K_{t+1}, h_t, M_t\} \\ \{\mathcal{Q}_t^H\}_{t=0}^\infty &= \{Y_t, X_t(i), K_t(i), h_t(i); i \in (0,1)\}_{t=0}^\infty\end{aligned}$$

such that:

(i) given a sequence of prices $\{\mathcal{P}_t\}_{t=0}^\infty$ *and a sequence of shocks,* $\{\mathcal{Q}_t^H\}_{t=0}^\infty$ *is a solution to the representative household's problem;*

(ii) given a sequence of prices $\{\mathcal{P}_t\}_{t=0}^\infty$ *and a sequence of shocks,* $\{\mathcal{Q}_t^F\}_{t=0}^\infty$ *is a solution to the representative firms' problem;*

(iii) given a sequence of quantities $\{\mathcal{Q}_t\}_{t=0}^\infty$ *and a sequence of shocks,* $\{\mathcal{P}_t\}_{t=0}^\infty$ *clears the markets*

$$Y_t = C_t + I_t + G_t \tag{14}$$

$$h_t = \int_0^1 h_t(i)\,di \tag{15}$$

$$K_t = \int_0^1 K_t(i)\,di \tag{16}$$

$$G_t = T_t \tag{17}$$

and the money market.

(iv) Prices satisfy (10) and (11).

2 Parametrization

The model is parameterized on US quarterly data for the post WWII period. The data are taken from the Federal Reserve Database.[5] The parameters are reported in table 1.

β, the discount factor is set such that households discount the future at a 4% annual rate, implying β equals 0.988. The instantaneous utility function takes the form

$$U\left(C_t, \frac{M_t}{P_t}, \ell_t\right) = \frac{1}{1-\sigma}\left[\left(\left(C_t^{\eta} + \zeta \frac{M_t}{P_t}^{\eta}\right)^{\frac{\nu}{\eta}} \ell_t^{1-\nu}\right)^{1-\sigma} - 1\right]$$

where ζ capture the preference for money holdings of the household. σ, the coefficient ruling risk aversion, is set equal to 1.5. ν is set such that the model generates a total fraction of time devoted to market activities of 31%. η is borrowed from Chari et al. (2000), who estimated it on postwar US data (-1.56). The value of ζ, 0.0649, is selected such that the model reproduces the average ratio of M1 money to nominal consumption expenditures.

γ, the probability of price resetting is set in the benchmark case at 0.25, implying that the average length of price contracts is about 4 quarters. The nominal growth of the economy, $\overline{\mu}$, is set such that the average quarterly rate of inflation over the period is $\overline{\pi} = 1.2\%$ per quarter. The quarterly depreciation rate, δ, is set equal to 0.025. θ in the benchmark case is set such that the level of markup in the steady state is 15%. α, the elasticity of the production function to physical capital, is set such that the model reproduces the US labor share — defined as the ratio of labor compensation to GDP — during the sample period (0.575).

The stochastic technology shock, $a_t = \log(A_t/\overline{A})$, is assumed to follow a stationary AR(1) process of the form

$$a_t = \rho_a a_{t-1} + \varepsilon_{a,t}$$

with $|\rho_a| < 1$ and $\varepsilon_{a,t} \rightsquigarrow \mathcal{N}(0, \sigma_a^2)$. We set $\rho_a = 0.95$ and $\sigma_a = 0.008$.

The government spending shock[6] is assumed to follow an AR(1) process

$$\log(g_t) = \rho_g \log(g_{t-1}) + (1-\rho_g)\log(\overline{g}) + \varepsilon_{g,t}$$

[5] URL: `http://research.stlouisfed.org/fred/`

[6] The –logarithm of the– government expenditure series is first detrended using a linear trend.

Table 1: Calibration: Benchmark case

Preferences		
Discount factor	β	0.988
Relative risk aversion	σ	1.500
Parameter of CES in utility function	η	-1.560
Weight of money in the utility function	ζ	0.065
CES weight in utility function	ν	0.344
Technology		
Capital elasticity of intermediate output	α	0.281
Capital adjustment costs parameter	φ	2.000
Depreciation rate	δ	0.025
Parameter of markup	θ	0.850
Probability of price resetting	γ	0.250
Shocks and policy parameters		
Persistence of technology shock	ρ_a	0.950
Standard deviation of technology shock	σ_a	0.008
Persistence of government spending shock	ρ_g	0.970
Volatility of government spending shock	σ_g	0.020
Persistence of money growth	ρ_m	0.500
Volatility of money shock	σ_m	0.007
Steady state money supply growth (gross)	μ	1.012
Inflation coefficient in Taylor rule	k_p	1.500
Output gap coefficient in Taylor rule	k_y	0.150
Persistence in interest rate rule	ρ	0.750
Share of government spending	g/y	0.200

with $|\rho_g| < 1$ and $\varepsilon_{g,t} \sim \mathcal{N}(0, \sigma_g^2)$. The persistence parameter is set to, ρ_g, of 0.97 and the standard deviation of innovations is $\sigma_g = 0.02$. The government spending to output ratio is set to 0.20.

When the monetary authorities are assumed to use an exogenous money supply rule, gross money growth evolves according to

$$\mu_t = (1 - \rho_m)\overline{\mu} + \rho_m \mu_{t-1} + \epsilon_{mt}$$

where $|\rho_m| < 1$, $\overline{\mu} = E(\mu_t)$ and ϵ_{mt} is a gaussian white noise process.

When an HTM rule is assumed, we use the values of $\rho_r = 0.75$, $k_\pi = 1.5$ and $k_y = 0.15$ suggested by Clarida, Gali and Gertler, 2000.

In order to investigate the role of information imperfection, we will assume that *some* of the variables are observed with error by the agents. In particular, for mis–measured variable x

$$x_t^\star = x_t^{\mathrm{T}} + \xi_t$$

where x_t^{T} denotes the value of the variable under perfect information and ξ_t is a noisy process that satisfies $E(\xi_t) = 0$ for all t; $E(\xi_t \varepsilon_{a,t}) = E(\xi_t \varepsilon_{g,t}) = 0$; and

$$E(\xi_t \xi_k) = \begin{cases} \sigma_\xi^2 & \text{if } t = k \\ 0 & \text{Otherwise} \end{cases}$$

In the version with exogenous money, we assume symmetric information for the government and the private agents. Their learning is based on the Kalman filter. Under the interest rate policy rule, there is asymmetric information regarding potential output (the government does not observe it).

In order to facilitate the interpretation of σ_ξ we set its value in relation to the volatility of the technology shock. More precisely, we define ς as $\varsigma = \sigma_\xi / \sigma_a$. We experiment with different values but end up reporting results with two values of ς, namely, $\varsigma = \{1, 3\}$. The latter value is close to that used elsewhere in the literature, for instance by Woodford, 2002.

One expects that the choice of the noisy variables would affect the properties of the model. While some variables are more likely to be observed with error than others, there is nothing in the literature that could help us operationalize the incidence and degree of mis-measurement. In the analysis below we have used alternative information sets and noisy variables. Encouragingly, the results are quite robust with regard to the location of the noise.

3 The results

The model is log–linearized around its deterministic steady state and solved according to the method detailed in appendix B. We first discuss the dynamic properties of the model following a monetary shock (in particular of inflation, output and the nominal interest rate) and then turn to its general properties.

3.1 The Transmission of Monetary Shocks

In this section we analyze the transmission of money supply shocks i n our benchmark economy. Hence, monetary authorities are assumed to supply money according to the exogenous rule (12). The results reported below have been obtained under the assumption that actual output and the technology shock are measured with error. We later present evidence that the results are quite robust to the choice of the noisy variables.

Figure 1 presents the response of inflation, output, the nominal and the real interest rate to a 1% shock to the growth rate of the money supply under perfect and imperfect information.[7] These figures confirm the well known fact that price staggering does not suffice to make the standard version of the new Keynesian model produce plausible dynamics. As can be seen from panel (a) of the figure, under perfect information, the response of inflation is monotonic and lacks persistence. Furthermore, the model does not generate a liquidity effect.

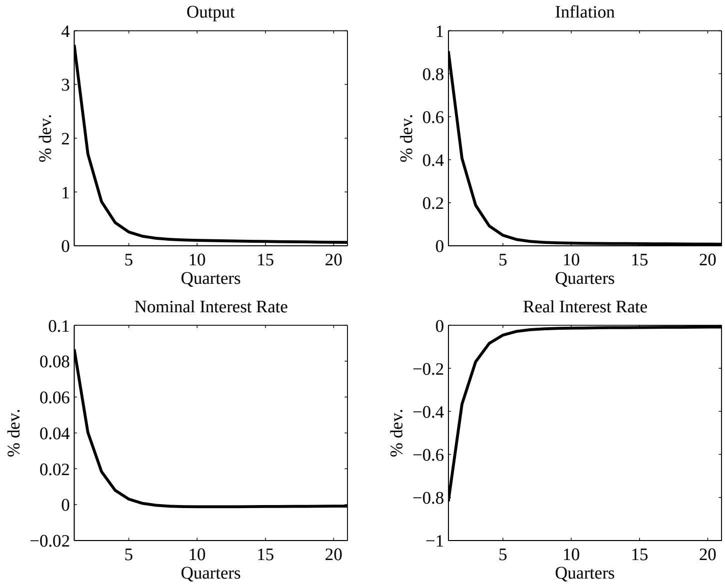
The results are considerably more realistic under imperfect information (panel (b)). Several features are worth noting. First, the model can generate persistent, hump–shaped dynamics for inflation and output. The degree of persistence depends on the degree of noise, with more noise translating into more persistence (compare to figure[8] 2 which uses both a lower ($\varsigma = 1$) and a higher ($\varsigma = 10$) degree of noise.). The model predicts a peak effect on inflation (and output) that occurs somewhat sooner than that in the real world (see Christiano, Eichenbaum and Evans, 2005). While this peak could be pushed further out by selecting a higher degree of imperfect information we are reluctant to use a much higher value because of two reasons. First, such a value is more likely to be perceived as implausibly high (in spite of the fact that we do not really know what constitutes a plausible degree of noise and what does not). And second and more important, larger values for noise create a trade off in performance, as they improve the dynamic paths but at the expense of a worse overall model fit, as will be shown later.

Second, inflation and economic activity are strongly positively correlated. While inflation and output peak at the same time, unlike in the data, where output peaks slightly ahead of inflation

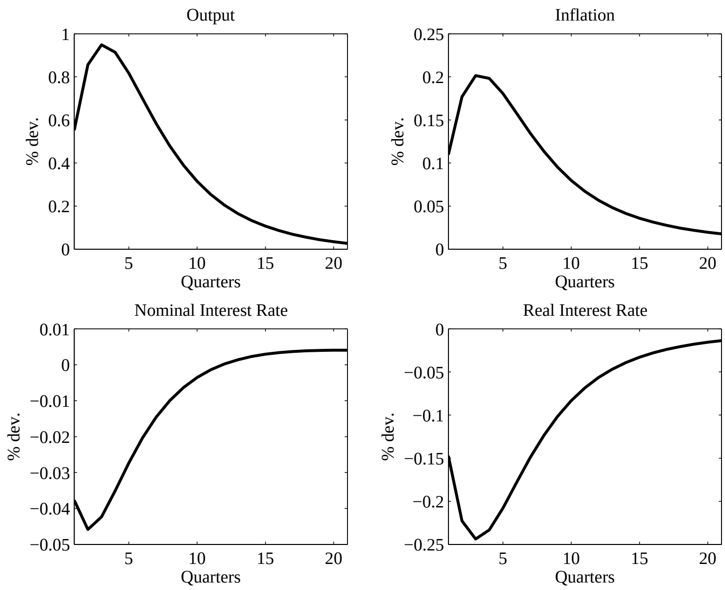
[7] Figure 4 in the appendix reports the IRFs with regard to a supply and fiscal shock.
[8] The figures for the sensitivity analysis are reported in appendix A.

Figure 1: IRF to a money supply shock

(a) Perfect Information



(b) Imperfect Information ($\zeta = 3$) with signals (y, a)



(Christiano, Eichenbaum and Evans (2005)) the path of output is steeper than that of inflation, that is, inflation is more inertial than economic activity. Moreover, an acceleration phenomenon obtains. When economic activity is strong, inflation tends to rise. Third, both the ex–post real and the *nominal* interest rate decrease persistently following a positive innovation in money growth, hence the model has no difficulty generating a liquidity effect. This is an important finding because no other version of the NK model has been able to capture this effect. And fourth, inflation forecast errors are serially correlated.

We have also examined whether the choice of the variable(s) that are subject to measurement error plays a critical role. It turns out that it does not. Namely, the hump shape in inflation and output as well as the timing patterns obtain even under alternative assumptions about the noisy variables. Likewise, the liquidity effect survives a change in the signals. This is shown in figure 3 which records, in panel (a), the IRF of inflation and output to a money shock under the assumption that output and inflation are measured with error, and in panel (b) which uses capital and employment as signals.

3.2 Business Cycle Properties

The ability to capture the dynamics of inflation, output and nominal interest rates is an important accomplishment. But it is not sufficient to establish the superiority of the imperfect information version over its perfect information counterpart. We now carry out a more general evaluation that is based on standard moment criteria. We report volatility, procyclicality and persistence for the key variables under alternative assumptions about the degree of imperfect information and the location of mis-measurement in the economy. In order to have a more realistic representation of monetary policy in the model we follow standard practice and adopt the interest policy rule (2) à la Henderson–McKibbin–Taylor described in section 1.4.

The behavior of the model as a function of imperfect information is reported in table 2. Panel (a) reports standard HP–filtered second order moments of the main macroeconomic variables under perfect information. Panel (b) records the same moments obtained when agents are imperfectly informed. As can be seen from the table, the results favor the imperfect information version. The main improvement is to be found in the volatility and cyclicality of inflation and of the interest rates. The imperfect information version can generate a volatility of hours worked that is greater than that of output without the need for indivisible labor, whereas the full information version of the model yields a volatility of hours of about half of that of output. The volatility of nominal interest rate is twice that of the full information model, but it still remains too low compared to the data (0.11 with $\zeta = 10$ against 0.40 in the data). Of particular interest is the behavior of inflation. Inflation volatility is magnified under imperfect information, as its

Table 2: HP–filtered second order moments

Var.	Std	Rel. Std	$\rho(\cdot, y)$	$\rho(1)$	$\rho(2)$
			Data		
y	1.49	1.00	1.00	0.88	0.70
c	0.80	0.54	0.86	0.87	0.69
i	6.03	4.04	0.92	0.83	0.61
h	1.88	1.26	0.83	0.92	0.73
π	0.16	0.11	0.32	0.33	0.24
R_{nom}	0.40	0.27	0.21	0.81	0.57
R_{real}	0.33	0.22	0.10	0.73	0.50
			(a) Perfect Information		
y	1.38	1.00	1.00	0.69	0.45
c	0.87	0.63	0.80	0.76	0.53
i	4.55	3.30	0.96	0.68	0.44
h	0.70	0.51	0.77	0.55	0.31
π	0.04	0.03	0.07	0.46	0.24
R_{nom}	0.03	0.02	0.21	0.89	0.71
R_{real}	0.03	0.02	0.13	0.35	0.12
			(b) Imperfect Information (ζ=3)		
y	1.41	1.00	1.00	0.68	0.44
c	0.86	0.61	0.90	0.69	0.45
i	4.45	3.16	0.88	0.68	0.44
h	1.49	1.06	0.68	0.40	0.13
π	0.16	0.11	0.42	0.33	0.06
R_{nom}	0.07	0.05	0.16	0.77	0.46
R_{real}	0.03	0.02	-0.21	0.82	0.62

Note: All series are HP–filtered. Data cover the period 1960:1–2002:4, except for aggregate weekly hours that run from 1964:1 to 2002:4. Output is defined as $C+I+G$. C is nondurables and services, I includes investment and durables. π is the CPI based inflation rate, R_{nom} is the federal fund rate, and $R_{real} = R_{nom} - \pi$. Std. is standard deviation, Rel. Std is standard deviation of the variable relative to that of output, $\rho(\cdot, y)$ is its correlation with output and $\rho(1)$ and $\rho(2)$ the first and second order autocorrelation.

standard deviation (0.16) is four times greater than that under perfect information. Here again, the imperfect information model outperforms the perfect information model as the data suggest a standard deviation of inflation around 0.16. Likewise, under imperfect information, inflation is much more procyclical than under perfect information, and the output–inflation correlation comes closer to the data. For instance, this correlation is about 0 under perfect information and 0.42 under imperfect information (ζ=3) while it is 0.3 in the data.

Table 3: HP–filtered second order moments (Sensitivity analysis)

Var.	Std	Rel. Std	$\rho(\cdot,y)$	$\rho(1)$	$\rho(2)$
	(a) Imperfect Information (ζ=1)				
y	1.44	1.00	1.00	0.68	0.43
c	0.89	0.62	0.88	0.72	0.49
i	4.58	3.18	0.90	0.68	0.43
h	1.15	0.80	0.70	0.43	0.19
π	0.11	0.07	0.33	0.30	0.11
R_{nom}	0.05	0.03	0.24	0.79	0.55
R_{real}	0.03	0.02	-0.38	0.83	0.63
	(b) Imperfect Information (ζ=10)				
y	1.18	1.00	1.00	0.66	0.42
c	0.77	0.65	0.85	0.67	0.43
i	3.59	3.05	0.82	0.68	0.46
h	1.66	1.41	0.60	0.48	0.21
π	0.20	0.17	0.39	0.48	0.20
R_{nom}	0.11	0.09	0.05	0.85	0.60
R_{real}	0.02	0.02	-0.10	0.81	0.60

Note: All series are HP–filtered. Std. is standard deviation, Rel. Std is standard deviation of the variable relative to that of output, $\rho(\cdot,y)$ is its correlation with output and $\rho(1)$ and $\rho(2)$ the first and second order autocorrelation.

Table 3 illustrates the role of information imperfection in the overall behavior of the economy, and makes it clear that the greater the degree of imperfection, the larger the volatility of nominal aggregates. In particular, the volatility of the nominal interest rate comes closer to that in the data, although it remains too low (0.11 with $\zeta = 10$ against 0.40 in the data).

Additional features, such as habit persistence, variable capitalization etc. can be easily added to the model in order to further improve its empirical performance, Nonetheless, this goes beyond our simple objective of examining the performance of the new Keynesian model as a function of the degree of imperfect information that is present in the decision of the agents and the policymakers.

4 Conclusions

The new Keynesian model has provided a popular framework for the analysis of monetary policy. Nevertheless, in spite of its overall success, the model has had difficulties accounting for the empirical behavior of key monetary variables such as inflation and nominal interest rates.

In this paper we have argued that there is no need to abandon the rational expectations assumption or to introduce other types of irrational behavior in order to help the model fit the data. A plausible assumption, namely that the true values of some macroeconomic variables are imperfectly observed, together with gradual learning suffice to resolve the empirical difficulties of the new Keynesian model. With a modest amount of imperfect information, both inflation and output dynamics follow hump shaped paths, inflation in persistent, a money shock has a liquidity effect and inflation expectations errors are serially correlated.

There exists another rational expectations version of the NK model that has similar properties, namely the sticky information model of Mankiw and Reis (2002). A fruitful course of research would be to undertake a systematic comparison of the imperfect information and sticky information versions of the NK in order to determine which variant –if any– fits the data better and which one seems more plausible. A comparison of the rational expectation versions of the NK model to those of limited rationality (such as Christiano et al. 2005) seems also a fruitful exercise (as in Collard and Dellas, 2005).

References

Bullard, James and Stefano Eusepi, 2003, "Did the Great Inflation Occur Despite Policymaker Commitment to a Taylor Rule." Federal Reserve Bank of Atlanta, October, WP 2003-20.

Christiano, L., C. Evans and M. Eichenbaum, 2005, Nominal Rigidities and the Dynamic Effects of a Shock to Monetary Policy, *Journal of Political Economy*, 113(1): 1–45.

Collard, Fabrice and Harris Dellas, 2004, "The Great Inflation of the 70s." ECB working paper, no.336.

Collard, Fabrice and Harris Dellas, 2005, "Dissecting the New Keynesian Model. " Mimeo.

Dellas, Harris, 2004, "Monetary shocks and inflation dynamics in the new Keynesian model." Mimeo.

Erceg, Christopher and Andrew Levin, 2003, "Imperfect Credibility and Inflation Persistence." Journal of Monetary Economics, 50(4), 915-944.

Gali, Jordi. 2003. New Perspectives on Monetary Policy, Inflation, and the Business Cycle, in Advances in Economic Theory, edited by: M. Dewatripont, L. Hansen, and S. Turnovsky, vol. III, 151-197, Cambridge University Press 2003.

Gali, Jordi and Mark Gertler, 1999, "Inflation Dynamics: A Structural Econometric Analysis.", Journal of Monetary Economics, 44(2), 195-222.

Ireland, Peter, 2000, "Expectations, Credibility, and Time-Consistent Monetary Policy.", Macroeconomic Dynamics 4, 448-466.

Mankiw, Gregory, and Ricardo Reis, 2002, "Sticky Information versus Sticky Prices: A Proposal to Replace the New Keynesian Phillips Curve." Quarterly Journal of Economics, 1295–1328.

Minford, Patrick and David, Peel, 2004, "Calvo Contracts: A Critique", CEPR WP 4288.

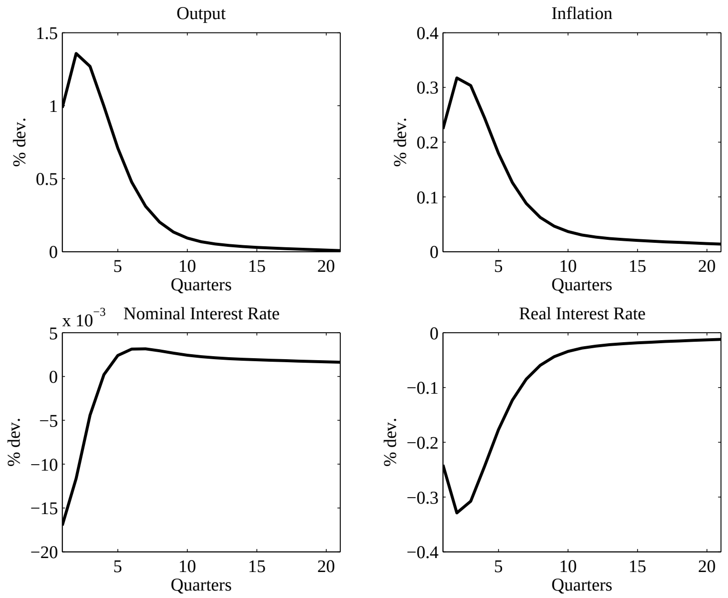
Roberts, John, 2001, How Well Does the New Keynesian Sticky-Price Model Fit the Data? Finance and Economics Discussion Paper no. 2001-13. Washington, D.C.: Board of Governors of the Federal Reserve System.

Woodford, Michael, 2002, "Imperfect Common Knowledge and the Effects of Monetary Policy.", in P. Aghion, R. Frydman, J. Stiglitz, and M. Woodford, eds., Knowledge, Information, and Expectations in Modern Macroeconomics: In Honor of Edmund S. Phelps, Princeton Univ. Press.

A Sensitivity Analysis

Figure 2: IRF to a money supply shock: Sensitivity to the amount of noise

(a) Low degree of imperfect information ($\varsigma = 1$)



(b) High degree of imperfect information ($\varsigma = 10$)

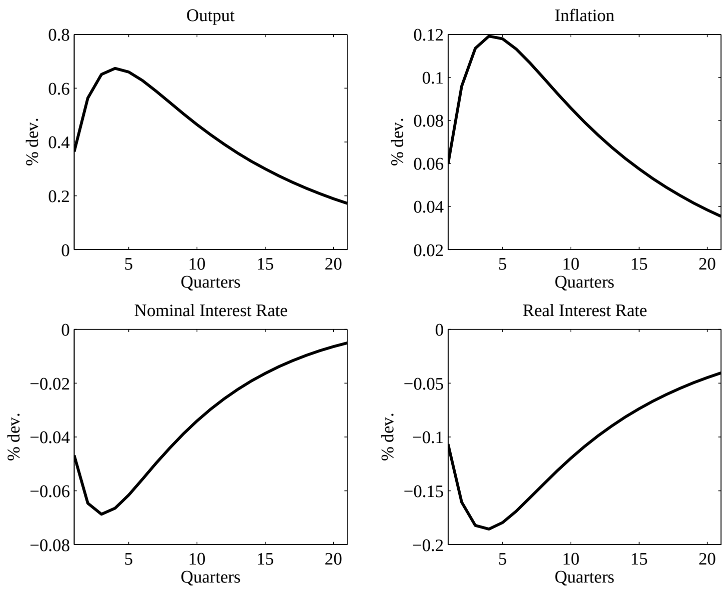
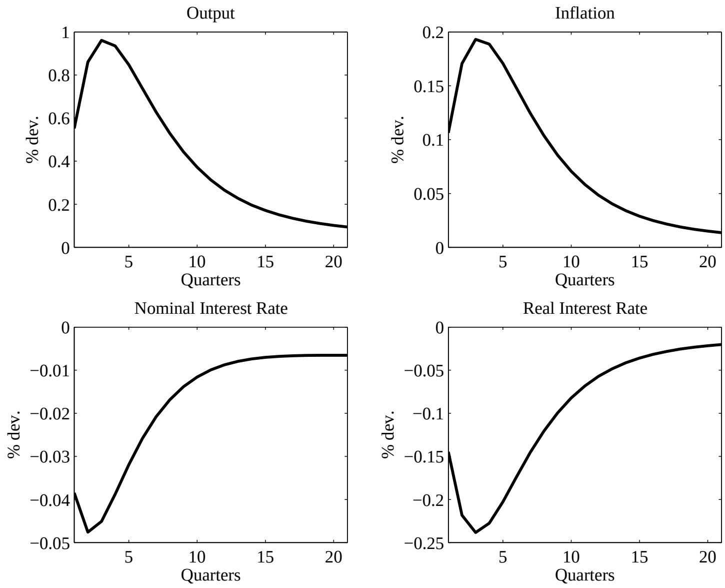


Figure 3: IRF to a money supply shock: Sensitivity to choice of signals ($\varsigma = 3$)

(a) Signals: (y,π)



(b) Signals: (k,h)

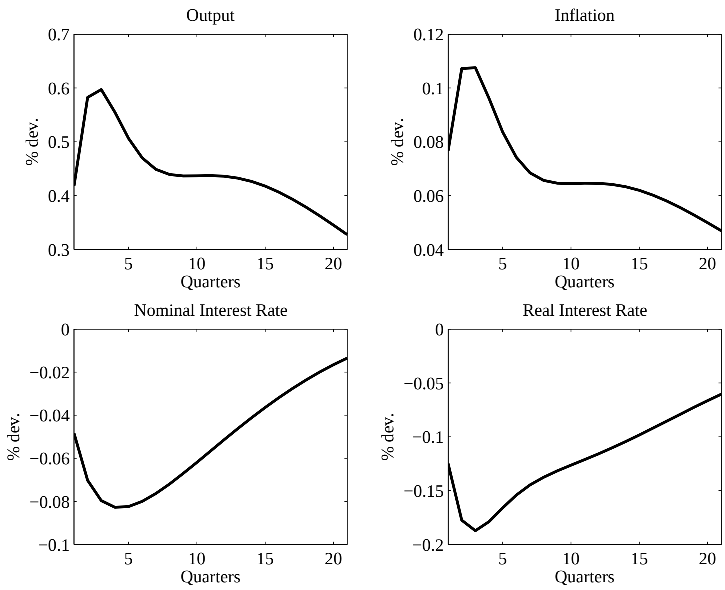
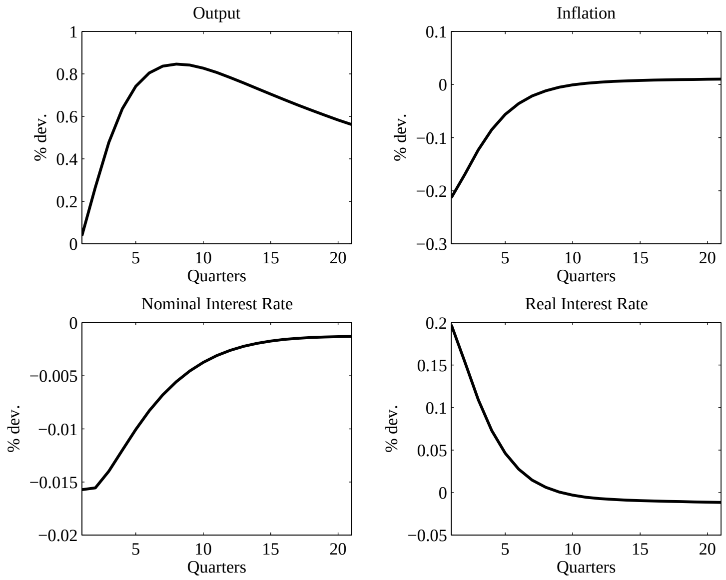
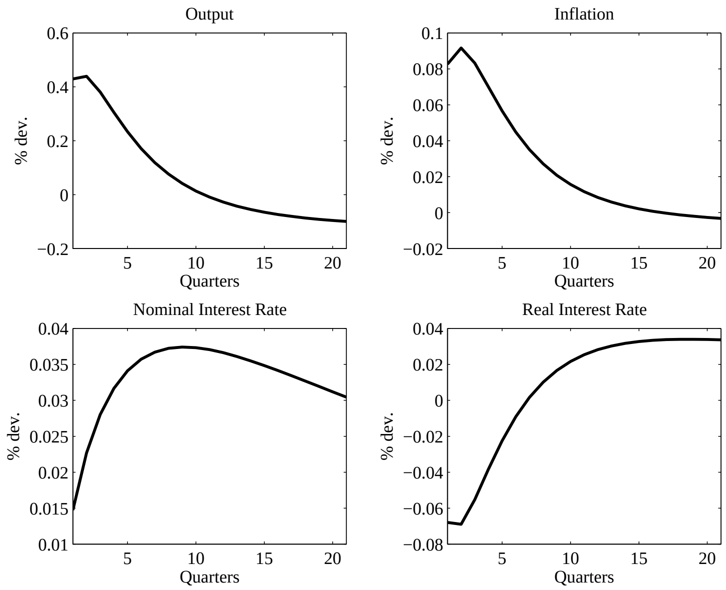


Figure 4: IRF to other shocks: Imperfect information ($\varsigma = 3$, signals=(y, a))

(a) Technology shocks



(b) Fiscal shocks



B Solution Method

The log–linear version of the system of dynamic equation characterizing the equilibrium may be written as

$$M_{cc}Y_t = M_{cs}\begin{pmatrix} X_t^b \\ X_t^f \end{pmatrix} + M_{ce}\begin{pmatrix} X_{t|t}^b \\ X_{t|t}^f \end{pmatrix} \tag{B.1}$$

$$M_{ss0}\begin{pmatrix} X_{t+1}^b \\ X_{t+1|t}^f \end{pmatrix} + M_{ss1}\begin{pmatrix} X_t^b \\ X_t^f \end{pmatrix} + M_{se1}\begin{pmatrix} X_{t|t}^b \\ X_{t|t}^f \end{pmatrix} = M_{sc0}Y_{t+1|t} + M_{sc1}Y_t + \begin{pmatrix} M_e u_{t+1} \\ 0 \end{pmatrix} \tag{B.2}$$

$$S_t = C^0\begin{pmatrix} X_t^b \\ X_t^f \end{pmatrix} + C^1\begin{pmatrix} X_{t|t}^b \\ X_{t|t}^f \end{pmatrix} + v_t \tag{B.3}$$

Y is a vector of n_y control variables, S is a vector of n_s signals used by the agents to form expectations, X^b is a vector of n_b predetermined (backward looking) state variables (including shocks to fundamentals), X^f is a vector of n_f forward looking state variables, finally u and v are two Gaussian white noise processes with variance–covariance matrices Σ_{uu} and Σ_{vv} respectively and $E(uv') = 0$.

$X_{t+i|t} = E(X_{t+i}|\mathcal{I}_t)$ for $i \geqslant 0$ and where $\mathcal{I}_t$ denotes the information set available to the agents at the beginning of period t. The information set available to the agents consists of *i)* the structure of the model and *ii)* the history of the observable signals they are given in each period:

$$\mathcal{I}_t = \{S_{t-j}, j \geqslant 0, M_{cc}, M_{cs}, M_{ce}, M_{ss0}, M_{ss1}, M_{sc0}, M_{sc1}, M_{se1}, M_e, C^0, C^1, \Sigma_{uu}, \Sigma_{vv}\}$$

Therefore, it is when we specify the signals that we may impose the information structure of the agents.

Before solving the system, note that, from (B.1), we have

$$Y_t = B^0\begin{pmatrix} X_t^b \\ X_t^f \end{pmatrix} + B^1\begin{pmatrix} X_{t|t}^b \\ X_{t|t}^f \end{pmatrix} \tag{B.4}$$

where $B^0 = M_{cc}^{-1}M_{cs}$ and $B^1 = M_{cc}^{-1}M_{ce}$, such that

$$Y_{t|t} = B\begin{pmatrix} X_{t|t}^b \\ X_{t|t}^f \end{pmatrix} \tag{B.5}$$

with $B = B^0 + B^1$.

B.1 Solving the system

Step 1: We first solve for the expected system:

$$M_{ss0}\begin{pmatrix} X^b_{t+1|t} \\ X^f_{t+1|t} \end{pmatrix} + (M_{ss1}+M_{se1})\begin{pmatrix} X^b_{t|t} \\ X^f_{t|t} \end{pmatrix} = M_{sc0}Y_{t+1|t} + M_{sc1}Y_{t|t} \tag{B.6}$$

Plugging (B.5) in (B.6), we get

$$\begin{pmatrix} X^b_{t+1|t} \\ X^f_{t+1|t} \end{pmatrix} = W\begin{pmatrix} X^b_{t|t} \\ X^f_{t|t} \end{pmatrix} \tag{B.7}$$

where

$$W = -\left(M_{ss0} - M_{sc0}B\right)^{-1}\left(M_{ss1} + M_{se1} - M_{sc1}B\right)$$

After getting the Jordan form associated to (B.7) and applying standard methods for eliminating bubbles, we get

$$X^f_{t|t} = GX^b_{t|t}$$

¿From which we get

$$X^b_{t+1|t} = (W_{bb} + W_{bf}G)X^b_{t|t} = W^b X^b_{t|t} \tag{B.8}$$

$$X^f_{t+1|t} = (W_{fb} + W_{ff}G)X^b_{t|t} = W^f X^b_{t|t} \tag{B.9}$$

Step 2: We go back to the initial system to get and write

Then, (B.2) rewrites

$$\begin{aligned} M_{ss0}\begin{pmatrix} X^b_{t+1} \\ X^f_{t+1|t} \end{pmatrix} + M_{ss1}\begin{pmatrix} X^b_{t} \\ X^f_{t} \end{pmatrix} + M_{se1}\begin{pmatrix} X^b_{t|t} \\ X^f_{t|t} \end{pmatrix} &= M_{sc0}B\begin{pmatrix} X^b_{t+1|t} \\ X^f_{t+1|t} \end{pmatrix} + M_{sc1}B^0\begin{pmatrix} X^b_{t} \\ X^f_{t} \end{pmatrix} \\ &+ M_{sc1}B^1\begin{pmatrix} X^b_{t|t} \\ X^f_{t|t} \end{pmatrix} + \begin{pmatrix} M_e u_{t+1} \\ 0 \end{pmatrix} \end{aligned}$$

Taking expectations, we have

$$\begin{aligned} M_{ss0}\begin{pmatrix} X^b_{t+1|t} \\ X^f_{t+1|t} \end{pmatrix} + M_{ss1}\begin{pmatrix} X^b_{t|t} \\ X^f_{t|t} \end{pmatrix} + M_{se1}\begin{pmatrix} X^b_{t|t} \\ X^f_{t|t} \end{pmatrix} &= M_{sc0}B\begin{pmatrix} X^b_{t+1|t} \\ X^f_{t+1|t} \end{pmatrix} + M_{sc1}B^0\begin{pmatrix} X^b_{t|t} \\ X^f_{t|t} \end{pmatrix} \\ &+ M_{sc1}B^1\begin{pmatrix} X^b_{t|t} \\ X^f_{t|t} \end{pmatrix} \end{aligned}$$

Substracting, we get

$$M_{ss0}\begin{pmatrix} X^b_{t+1} - X^b_{t+1|t} \\ 0 \end{pmatrix} + M_{ss1}\begin{pmatrix} X^b_{t} - X^b_{t|t} \\ X^f_{t} - X^f_{t|t} \end{pmatrix} = M_{sc1}B^0\begin{pmatrix} X^b_{t} - X^b_{t|t} \\ X^f_{t} - X^f_{t|t} \end{pmatrix} + \begin{pmatrix} M_e u_{t+1} \\ 0 \end{pmatrix} \tag{B.10}$$

which rewrites

$$\begin{pmatrix} X^b_{t+1} - X^b_{t+1|t} \\ 0 \end{pmatrix} = W^c \begin{pmatrix} X^b_t - X^b_{t|t} \\ X^f_t - X^f_{t|t} \end{pmatrix} + M^{-1}_{ss0} \begin{pmatrix} M_e u_{t+1} \\ 0 \end{pmatrix} \tag{B.11}$$

where, $W^c = -M^{-1}_{ss0}(M_{ss1} - M_{sc1}B^0)$. Hence, considering the second block of the above matrix equation, we get

$$W^c_{fb}(X^b_t - X^b_{t|t}) + W^c_{ff}(X^f_t - X^f_{t|t}) = 0$$

which gives

$$X^f_t = F^0 X^b_t + F^1 X^b_{t|t}$$

with $F^0 = -{W^c_{ff}}^{-1} W^c_{fb}$ and $F^1 = G - F^0$.

Now considering the first block we have

$$X^b_{t+1} = X^b_{t+1|t} + W^c_{bb}(X^b_t - X^b_{t|t}) + W^c_{bf}(X^f_t - X^f_{t|t}) + M^2 u_{t+1}$$

from which we get, using (B.8)

$$X^b_{t+1} = M^0 X^b_t + M^1 X^b_{t|t} + M^2 u_{t+1}$$

with $M^0 = W^c_{bb} + W^c_{bf} F^0$, $M^1 = W^b - M^0$ and $M^2 = M^{-1}_{ss0} M_e$..

We also have

$$S_t = C^0_b X^b_t + C^0_t X^f_t + C^1_b X^b_{t|t} + C^1_f X^f_{t|t} + v_t$$

from which we get

$$S_t = S^0 X^b_t + S^1 X^b_{t|t} + v_t$$

where $S^0 = C^0_b + C^0_f F^0$ and $S^1 = C^1_b + C^0_f F^1 + C^1_f G$

Finally, we get

$$Y_t = B^0_b X^b_t + B^0_t X^f_t + B^1_b X^b_{t|t} + B^1_f X^f_{t|t}$$

from which we get

$$Y_t = \Pi^0 X^b_t + \Pi^1 X^b_{t|t}$$

where $\Pi^0 = B^0_b + B^0_f F^0$ and $\Pi^1 = B^1_b + B^0_f F^1 + B^1_f G$

B.2 Filtering

Since our solution involves terms in $X^b_{t|t}$, we would like to compute this quantity. However, the only information we can exploit is a signal S_t that we described previously. We therefore use a Kalman filter approach to compute the optimal prediction of $X^b_{t|t}$.

In order to recover the Kalman filter, it is a good idea to think in terms of expectation errors. Therefore, let us define

$$\widehat{X}_t^b = X_t^b - X_{t|t-1}^b$$

and

$$\widehat{S}_t = S_t - S_{t|t-1}$$

Note that since S_t depends on $X_{t|t}^b$, only the signal relying on $\widetilde{S}_t = S_t - S^1 X_{t|t}^b$ can be used to infer anything on $X_{t|t}^b$. Therefore, the policy maker revises its expectations using a linear rule depending on $\widetilde{S}_t^e = S_t - S^1 X_{t|t}^b$. The filtering equation then writes

$$X_{t|t}^b = X_{t|t-1}^b + K(\widetilde{S}_t^e - \widetilde{S}_{t|t-1}^e) = X_{t|t-1}^b + K(S^0 \widehat{X}_t^b + v_t)$$

where K is the filter gain matrix, that we would like to compute.

The first thing we have to do is to rewrite the system in terms of state–space representation. Since $S_{t|t-1} = (S^0 + S^1) X_{t|t-1}^b$, we have

$$\begin{aligned}
\widehat{S}_t &= S^0(X_t^b - X_{t|t}^b) + S^1(X_{t|t}^b - X_{t|t-1}^b) + v_t \\
&= S^0 \widehat{X}_t^b + S^1 K(S^0 \widehat{X}_t^b + v_t) + v_t \\
&= S^\star \widehat{X}_t^b + \nu_t
\end{aligned}$$

where $S^\star = (I + S^1 K) S^0$ and $\nu_t = (I + S^1 K) v_t$.

Now, consider the law of motion of backward state variables, we get

$$\begin{aligned}
\widehat{X}_{t+1}^b &= M^0(X_t^b - X_{t|t}^b) + M^2 u_{t+1} \\
&= M^0(X_t^b - X_{t|t-1}^b - X_{t|t}^b + X_{t|t-1}^b) + M^2 u_{t+1} \\
&= M^0 \widehat{X}_t^b - M^0(X_{t|t}^b + X_{t|t-1}^b) + M^2 u_{t+1} \\
&= M^0 \widehat{X}_t^b - M^0 K(S^0 \widehat{X}_t^b + v_t) + M^2 u_{t+1} \\
&= M^\star \widehat{X}_t^b + \omega_{t+1}
\end{aligned}$$

where $M^\star = M^0(I - KS^0)$ and $\omega_{t+1} = M^2 u_{t+1} - M^0 K v_t$.

We therefore end–up with the following state–space representation

$$\widehat{X}_{t+1}^b = M^\star \widehat{X}_t^b + \omega_{t+1} \qquad \text{(B.12)}$$

$$\widehat{S}_t = S^\star \widehat{X}_t^b + \nu_t \qquad \text{(B.13)}$$

For which the Kalman filter is given by

$$\widehat{X}_{t|t}^b = \widehat{X}_{t|t-1}^b + PS^{\star\prime}(S^\star P S^{\star\prime} + \Sigma_{\nu\nu})^{-1}(S^\star \widehat{X}_t^b + \nu_t)$$

But since $\widehat{X}^b_{t|t}$ is an expectation error, it is not correlated with the information set in $t-1$, such that $\widehat{X}^b_{t|t-1} = 0$. The prediction formula for $\widehat{X}^b_{t|t}$ therefore reduces to

$$\widehat{X}^b_{t|t} = PS^{\star\prime}(S^\star PS^{\star\prime} + \Sigma_{\nu\nu})^{-1}(S^\star \widehat{X}^b_t + \nu_t) \tag{B.14}$$

where P solves

$$P = M^\star PM^{\star\prime} + \Sigma_{\omega\omega}$$

and $\Sigma_{\nu\nu} = (I + S^1K)\Sigma_{vv}(I + S^1K)'$ and $\Sigma_{\omega\omega} = M^0K\Sigma_{vv}K'M^{0\prime} + M^2\Sigma_{uu}M^{2\prime}$

Note however that the above solution is obtained for a given K matrix that remains to be computed. We can do that by using the basic equation of the Kalman filter:

$$\begin{aligned}
X^b_{t|t} &= X^b_{t|t-1} + K(\widetilde{S}^e_t - \widetilde{S}^e_{t|t-1}) \\
&= X^b_{t|t-1} + K(S_t - S^1X^b_{t|t} - (S_{t|t-1} - S^1X^b_{t|t-1})) \\
&= X^b_{t|t-1} + K(S_t - S^1X^b_{t|t} - S^0X^b_{t|t-1})
\end{aligned}$$

Solving for $X^b_{t|t}$, we get

$$\begin{aligned}
X^b_{t|t} &= (I + KS^1)^{-1}(X^b_{t|t-1} + K(S_t - S^0X^b_{t|t-1})) \\
&= (I + KS^1)^{-1}(X^b_{t|t-1} + KS^1X^b_{t|t-1} - KS^1X^b_{t|t-1} + K(S_t - S^0X^b_{t|t-1})) \\
&= (I + KS^1)^{-1}(I + KS^1)X^b_{t|t-1} + (I + KS^1)^{-1}K(S_t - (S^0 + S^1)X^b_{t|t-1})) \\
&= X^b_{t|t-1} + (I + KS^1)^{-1}K\widehat{S}_t \\
&= X^b_{t|t-1} + K(I + S^1K)^{-1}\widehat{S}_t \\
&= X^b_{t|t-1} + K(I + S^1K)^{-1}(S^\star\widehat{X}^b_t + \nu_t)
\end{aligned}$$

where we made use of the identity $(I+KS^1)^{-1}K \equiv K(I+S^1K)^{-1}$. Hence, identifying to (B.14), we have

$$K(I + S^1K)^{-1} = PS^{\star\prime}(S^\star PS^{\star\prime} + \Sigma_{\nu\nu})^{-1}$$

remembering that $S^\star = (I + S^1K)S^0$ and $\Sigma_{\nu\nu} = (I + S^1K)\Sigma_{vv}(I + S^1K)'$, we have

$$K(I+S^1K)^{-1} = PS^{0\prime}(I+S^1K)'((I+S^1K)S^0PS^{0\prime}(I+S^1K)'+(I+S^1K)\Sigma_{vv}(I+S^1K)')^{-1}(I+S^1K)S^0$$

which rewrites as

$$\begin{aligned}
K(I + S^1K)^{-1} &= PS^{0\prime}(I + S^1K)'\left[(I + S^1K)(S^0PS^{0\prime} + \Sigma_{vv})(I + S^1K)'\right]^{-1} \\
K(I + S^1K)^{-1} &= PS^{0\prime}(I + S^1K)'(I + S^1K)'^{-1}(S^0PS^{0\prime} + \Sigma_{vv})^{-1}(I + S^1K)^{-1}
\end{aligned}$$

Hence, we obtain

$$K = PS^{0\prime}(S^0PS^{0\prime} + \Sigma_{vv})^{-1} \tag{B.15}$$

Now, recall that

$$P = M^{\star} P M^{\star\prime} + \Sigma_{\omega\omega}$$

Remembering that $M^{\star} = M^0(I + KS^0)$ and $\Sigma_{\omega\omega} = M^0 K \Sigma_{vv} K' M^{0\prime} + M^2 \Sigma_{uu} M^{2\prime}$, we have

$$\begin{aligned} P &= M^0(I - KS^0)P\left[M^0(I - KS^0)\right]' + M^0 K \Sigma_{vv} K' M^{0\prime} + M^2 \Sigma_{uu} M^{2\prime} \\ &= M^0\left[(I - KS^0)P(I - S^{0\prime}K') + K\Sigma_{vv}K'\right] M^{0\prime} + M^2 \Sigma_{uu} M^{2\prime} \end{aligned}$$

Plugging the definition of K in the latter equation, we obtain

$$P = M^0\left[P - PS^{0\prime}(S^0 P S^{0\prime} + \Sigma_{vv})^{-1} S^0 P\right] M^{0\prime} + M^2 \Sigma_{uu} M^{2\prime} \tag{B.16}$$

B.3 Summary

We finally end–up with the system of equations:

$$X^b_{t+1} = M^0 X^b_t + M^1 X^b_{t|t} + M^2 u_{t+1} \tag{B.17}$$

$$S_t = S^0_b X^b_t + S^1_b X^b_{t|t} + v_t \tag{B.18}$$

$$Y_t = \Pi^0_b X^b_t + \Pi^1_b X^b_{t|t} \tag{B.19}$$

$$X^f_t = F^0 X^b_t + F^1 X^b_{t|t} \tag{B.20}$$

$$X^b_{t|t} = X^b_{t|t-1} + K(S^0(X^b_t - X^b_{t|t-1}) + v_t) \tag{B.21}$$

$$X^b_{t+1|t} = (M^0 + M^1) X^b_{t|t} \tag{B.22}$$

to describe the dynamics of our economy.

This may be recasted as a standard state–space problem as

$$\begin{aligned} X^b_{t+1|t+1} &= X^b_{t+1|t} + K(S^0(X^b_{t+1} - X^b_{t+1|t}) + v_{t+1}) \\ &= (M^0 + M^1)X^b_{t|t} + K(S^0(M^0 X^b_t + M^1 X^b_{t|t} + M^2 u_{t+1} - (M^0 + M^1)X^b_{t|t}) + v_{t+1}) \\ &= KS^0 M^0 X^b_t + ((I - KS^0)M^0 + M^1)X^b_{t|t} + KS^0 M^2 u_{t+1} + K v_{t+1} \end{aligned}$$

Then

$$\begin{pmatrix} X^b_{t+1} \\ X^b_{t+1|t+1} \end{pmatrix} = M_{\text{X}} \begin{pmatrix} X^b_t \\ X^b_{t|t} \end{pmatrix} + M_{\text{E}} \begin{pmatrix} u_{t+1} \\ v_{t+1} \end{pmatrix}$$

where

$$M_{\text{X}} = \begin{pmatrix} M^0 & M^1 \\ KS^0 M^0 & ((I - KS^0)M^0 + M^1) \end{pmatrix} \text{ and } M_{\text{E}} = \begin{pmatrix} M^2 & 0 \\ KS^0 M^2 & K \end{pmatrix}$$

and

$$Y_t = M_{\text{Y}} \begin{pmatrix} X^b_t \\ X^b_{t|t} \end{pmatrix}$$

where

$$M_{\text{Y}} = \begin{pmatrix} \Pi^0_b & \Pi^1_b \end{pmatrix}$$