Score-driven models for forecasting

F. Blasques  S.J. Koopman  A. Lucas
VU University Amsterdam, Tinbergen Institute, CREATES

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The basic framework

We have an observed time series \( y_1, \ldots, y_n \), collected in \( y \), and an unobserved path of equal length \( f_1, \ldots, f_n \), collected in \( f \).

Assume that

- DGP is \( y \sim p(y; f) \) where \( f \) represents a time-varying feature of the "true" model density.
- time series econometrician opts for a predictive model \( \tilde{p}(y_t|y_1, \ldots, y_{t-1}; \theta) \), with parameter vector \( \theta \), and correctly considers the effect associated with \( f \) to be time-varying.
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- econometric model \( \hat{p}(y_t|y_1, \ldots, y_{t-1}; \theta) \) correctly considers the effect associated with \( f \) to be time-varying.
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In an observation-driven model, the time-varying effect is extracted as a direct function of past data:

\[
\tilde{f}_t = f_t(y_1, \ldots, y_{t-1}; \theta) \quad \text{We have} \quad \tilde{p}(y_t|\tilde{f}_t, y_1, \ldots, y_{t-1}; \theta).
\]
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\tilde{f}_t = \tilde{f}_t(y_1, \ldots, y_{t-1}; \theta).
\]
We have \( \tilde{p}(y_t|\tilde{f}_t, y_1, \ldots, y_{t-1}; \theta) \).

Consider GARCH model where \( f \) is the time-varying variance, then we have \( \tilde{p}(y_t|\tilde{f}_t; \theta) \equiv N(0, \tilde{f}_t) \) and
\[
\tilde{f}_{t+1} = \omega + \beta \tilde{f}_t + \alpha (y_t^2 - \tilde{f}_t),
\]
and with \( \theta = (\omega, \alpha, \beta)' \).
The basic framework

In a general setting (Creal, Koopman & Lucas 2008, 2011, 2013),

\[ \tilde{p}(y_t|\tilde{f}_t, y_1, \ldots, y_{t-1}; \theta) \neq N(0, \tilde{f}_t), \]

we propose for \( \tilde{f}_t \) to adopt the updating scheme

\[ \tilde{f}_{t+1} = \omega + \beta \tilde{f}_t + \alpha s_t, \]

where \( s_t \) is the scaled score function \( s_t = S_t \nabla_t \) with

\[
\nabla_t = \frac{\partial \ln p(y_t|y_1, \ldots, y_{t-1}, \tilde{f}_t; \theta)}{\partial \tilde{f}_t},
\]

\[
S_t = \mathcal{L}_{t-1}^{-1} = -E_{t-1} \left[ \frac{\partial^2 \ln p(y_t|Y_{t-1}, \tilde{f}_t; \theta)}{\partial f_t \partial \tilde{f}_t'} \right]^{-1}.
\]
Score-driven models

For this updating equation

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Score-driven models

For this updating equation

\[ \tilde{f}_{t+1} = \omega + \beta \tilde{f}_t + \alpha s_t, \]

- we can view updating as a steepest ascent or Newton step for \( \tilde{f}_t \) using the log conditional density \( \tilde{p}(y_t|\tilde{f}_t, y_1, \ldots, y_{t-1}; \theta) \) as criterion function;
- choice for \( S_t \) may be square root matrix of the inverse Fisher information matrix;
- this \( S_t \) accounts for curvature of density as function of \( \tilde{f}_t \);
- also, under correct model specification, \( s_t \) has unit variance.
Generalized Autoregressive Score (GAS)

The general GAS framework is

\[ \tilde{p}(y_t | \tilde{f}_t, y_1, \ldots, y_{t-1}; \theta), \]

where for \( \tilde{f}_t \) we adopt the updating scheme

\[ \tilde{f}_{t+1} = \omega + \beta \tilde{f}_t + \alpha s_t, \]

with \( s_t \) as the scaled score function \( s_t = S_t \nabla_t \) and

\[ \nabla_t = \frac{\partial \ln p(y_t | y_1, \ldots, y_{t-1}, \tilde{f}_t; \theta)}{\partial \tilde{f}_t}, \]

\[ S_t = \mathcal{I}_{t-1}^{-1/2} = -E_{t-1} \left[ \frac{\partial^2 \ln p(y_t | Y_{t-1}, \tilde{f}_t; \theta)}{\partial \tilde{f}_t \partial \tilde{f}_t'} \right]^{-1/2}. \]
Illustration: volatility modeling

A class of volatility models is given by

\[ y_t = \mu + \sigma(\tilde{f}_t)u_t, \quad u_t \sim p_u(u_t; \theta), \]

\[ \tilde{f}_{t+1} = \omega + \beta \tilde{f}_t + \alpha s_t, \]

where:

- \( \sigma() \) is some continuous function;
- \( p_u(u_t; \theta) \) is a standardized disturbance density;
- \( s_t \) is the scaled score based on \( \partial \log p(y_t|\tilde{f}_t, y^{t-1}; \theta) / \partial \tilde{f}_t \).

Some special cases

- \( \sigma(\tilde{f}_t) = f_t \) and \( p_u \) is Gaussian: GAS \( \Rightarrow \) GARCH;
- \( \sigma(\tilde{f}_t) = \exp(\tilde{f}_t) \) and \( p_u \) is Gaussian: GAS \( \Rightarrow \) EGARCH;
- \( \sigma(\tilde{f}_t) = \exp(\tilde{f}_t) \) and \( p_u \) is Student’s t: GAS \( \Rightarrow \) t-GAS.
Special cases of GAS

GAS updating for specific observation densities and scaling choices reduces to well-known models.

- **GARCH** for $N(0, \tilde{f}_t)$: Engle (1982), Bollerslev (1986)
- **EGARCH** for $N(0, \exp \tilde{f}_t)$: Nelson (1991)
- Exponential distribution (ACD and ACI): Engle & Russell (1998) and Russell (2001), respectively
- Poisson d.: Davis, Dunsmuir & Street (2003)

New GAS applications: [http://gasmodel.com](http://gasmodel.com)
Why GAS?

- In econometrics, score and Hessian are familiar entities;
- Using contribution of score at time $t$ of updating seems not unreasonable (quasi-Newton connection);
- Many GARCH-type time series models are effectively constructed in this way.
- In case of GARCH, the score driver has an interpretation: $E_{t-1}(y_t^2) = \sigma^2$.
- In other cases (incl. GARCH with $t$-densities), choice of driver mechanism for updating is not so clear.
- But is the score then really a good idea?
Why GAS?

On the outset:

1. GAS update is very intuitive...
2. Using the score seems optimal in some sense...
3. Likelihood and KL divergence are closely related...

Question: Is the GAS update optimal in some KL sense?

Challenge 1: GAS update is surely not always correct!

Challenge 2: Comparing different observation-driven models is only interesting under misspecification with very general DGP!
DGP and Observation-Driven Model

True sequence of conditional densities:

\[ \{ p(y_t|f_t) \} , \quad \text{true tv parameter } \{ f_t \} \]

Conditional densities postulated by probabilistic model:

\[ \{ \tilde{p}(y_t|\tilde{f}_t; \theta) \} , \quad \text{filtered tv parameter } \{ \tilde{f}_t \} \]

\( \tilde{p}(y_t|\tilde{f}_t; \theta) \) is implicitly by observation equation:

\[ y_t = g(\tilde{f}_t, u_t; \theta), \quad u_t \sim p_u(\theta), \]

Observation-driven parameter update:

\[ \tilde{f}_{t+1} = \phi(y_t, \tilde{f}_t; \theta), \quad \forall \ t \in \mathbb{N}, \quad \tilde{f}_1 \in \mathcal{F} \subseteq \mathbb{R}, \]
**Key objective:** Characterize $\phi(\cdot)$ that possess optimality properties from information theoretic point of view.

**Main Question:** Is there an optimal form for the update

$$\tilde{f}_{t+1} = \phi(y_t, \tilde{f}_t; \theta), \quad \forall \ t \in \mathbb{N}, \quad \tilde{f}_1 \in \mathcal{F} \subseteq \mathbb{R},$$

**Answer:** This depends on the notion of optimality!

**Result 1:** Only parameter updates based on the score always reduce the local Kullback-Leibler divergence $p$ and $\tilde{p}$.

**Result 2:** The use of the score leads to considerably smaller global KL divergence in empirically relevant settings.

**Note:** Results hold for any DGP (any $p$ and $\{f_t\}$)
Definitions: Local GAS Updates

**GAS-update:**

\[ \tilde{f}_{t+1} = \phi(y_t, \tilde{f}_t; \theta) = \omega + \alpha s(y_t, \tilde{f}_t) + \beta \tilde{f}_t, \quad \forall \ t \in \mathbb{N}, \]

**Newton-GAS update:** ( \( \omega = 0, \alpha > 0, \beta = 1 \) )

\[ \tilde{f}_{t+1} = \alpha s(y_t, \tilde{f}_t) + \tilde{f}_t, \quad \forall \ t \in \mathbb{N}, \]

**Local update:** \( \tilde{f}_{t+1} \) in neighborhood of \( \tilde{f}_t \)

**Local optimality:** Refers to local updates
Definition I: Realized KL Divergence

KL divergence between \( p(\cdot|f_t) \) and \( \tilde{p}(\cdot|\tilde{f}_{t+1}; \theta) \) is given by

\[
D_{KL}(p(\cdot|f_t), \tilde{p}(\cdot|\tilde{f}_{t+1}; \theta)) = \int_{-\infty}^{\infty} p(y_t|f_t) \ln \frac{p(y_t|f_t)}{\tilde{p}(y_t|\tilde{f}_{t+1}; \theta)} \, dy_t.
\]
Definition I: Realized KL Divergence

**KL divergence** between $p(\cdot | f_t)$ and $\tilde{p}(\cdot | \tilde{f}_{t+1}; \theta)$ is given by

$$D_{KL}(p(\cdot | f_t), \tilde{p}(\cdot | \tilde{f}_{t+1}; \theta)) = \int_{-\infty}^{\infty} p(y_t | f_t) \ln \frac{p(y_t | f_t)}{\tilde{p}(y_t | \tilde{f}_{t+1}; \theta)} \, dy_t.$$ 

The **realized KL variation** $\Delta_{RKL}^{t-1}$ of a parameter update from $\tilde{f}_t$ to $\tilde{f}_{t+1}$ is defined as

$$\Delta_{RKL}^{t-1} = D_{KL}(p(\cdot | f), \tilde{p}(\cdot | \tilde{f}_t; \theta)) - D_{KL}(p(\cdot | f_t), \tilde{p}(\cdot | \tilde{f}_t; \theta)).$$
Definition II: Conditionally Expected KL Divergence

An optimal updating scheme, while subject to randomness, should have tendency to move in *correct direction*:

On average, the KL divergence should reduce in expectation.
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An optimal updating scheme, while subject to randomness, should have tendency to move in *correct direction*:

On average, the KL divergence should reduce in expectation.

The *conditionally expected KL (CKL)* variation of a parameter update from $\tilde{f}_t \in \tilde{F}$ to $\tilde{f}_{t+1} \in \tilde{F}$ is given by

$$
\Delta_{CKL}^{t-1} = \int_{F} q(\tilde{f}_{t+1}|\tilde{f}_t, f_t; \theta) \left[ \int_{Y} p(y|f_t) \ln \frac{\tilde{p}(y|\tilde{f}_t; \theta)}{\tilde{p}(y|\tilde{f}_{t+1}; \theta)} dy \right] d\tilde{f}_{t+1},
$$

where $q(\tilde{f}_{t+1}|\tilde{f}_t, f_t; \theta)$ denotes the density of $\tilde{f}_{t+1}$ conditional on both $\tilde{f}_t$ and $f_t$. For a given $p_t$, an update is CKL optimal if and only if $\Delta_{CKL}^{t-1} \leq 0$. 

Propositions 1 and 2

_all subject to some regularity conditions_

Proposition 1:

Every Newton-GAS update \( \tilde{f}_{t+1} = \alpha s_t + \tilde{f}_t \) is locally \textit{RKL optimal} and \textit{CKL optimal} for any true density \( p_t \).
Propositions 1 and 2

all subject to some regularity conditions

**Proposition 1:**

Every Newton-GAS update \((\tilde{f}_{t+1} = \alpha s_t + \tilde{f}_t)\) is locally \(RKL\) optimal and \(CKL\) optimal for any true density \(p_t\).

These properties can be generalized to ‘score-equivalent’ updates, we require a fundamental sign condition.
Propositions 1 and 2

all subject to some regularity conditions

Proposition 1:

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These properties can be generalized to ‘score-equivalent’ updates, we require a fundamental sign condition.

Proposition 2:

For any given true density $p_t$, a parameter update is locally RKL optimal and CKL optimal if and only if the parameter update is \textit{score-equivalent}, that is

$$\text{sign}(\Delta \phi(f, y; \theta)) = \text{sign}(\tilde{\nabla}(f, y; \theta))$$
Proposition 3

For the GAS updating

$$\tilde{f}_{t+1} = \omega + \beta \tilde{f}_t + \alpha s_t,$$

we require specific and different conditions for $\omega$, $\beta$ and $\alpha$ to state

*Proposition 3:*

The GAS update is locally RKL optimal and CKL optimal, for every $p_t$. 
The GAS volatility model with a normal distribution reduces to the standard GARCH model.

Then the necessary condition is $\alpha |y_t^2 - \tilde{f}_t| > |\omega + (\beta - 1)\tilde{f}_t| :$

It follows that the update becomes convincingly more RKL optimal if the observed $y_t^2$ deviates considerably from the filtered volatility $\tilde{f}_t$. 
Application: Volatility Model

Data Generating Process:

\[ y_t = \sqrt{f_t} u_t, \quad u_t \sim \tau(\lambda), \]
\[ \log f_t = a + b \log f_{t-1} + \epsilon_t, \quad \epsilon_t \sim \text{NID}(0, \sigma^2_\epsilon), \]

Parameter Values: \( a = 0, b = 0.98, \sigma_\epsilon = 0.065, \lambda \in [2, 8] \).

Compared Models: GARCH, t-GARCH and t-GAS.

\[ y_t = \sqrt{\tilde{f}_t} u_t \]
\[ (\text{GARCH}) \quad \tilde{f}_{t+1} = \omega + \alpha y_t^2 + \beta \tilde{f}_t, \quad u_t \sim N(0, \sigma^2) \]
\[ (\text{t-GARCH}) \quad \tilde{f}_{t+1} = \omega + \alpha y_t^2 + \beta \tilde{f}_t, \quad u_t \sim \tau(\nu) \]
\[ (\text{t-GAS}) \quad \tilde{f}_{t+1} = \omega + \alpha s(y_t, \tilde{f}_t) + \beta \tilde{f}_t, \quad u_t \sim \tau(\nu) \]
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Parameter Values: \( a = 0, b = 0.98, \sigma_\epsilon = 0.065, \lambda \in [2, 8] \).

Note: Comparison at pseudo-true parameters \( \theta^* = \arg \min \text{KL} \)

Sample size \( T \): Large enough for ML estimator to converge to pseudo-true parameter (at 3rd decimal place).

Asymptotic Theory: Convergence of ML estimator to pseudo-true parameter in misspecified GAS, see BKL (2013).
The pseudo-true parameters

Figure: DGP with $a = 0$, $b = 0.98$, $\sigma_\epsilon = 0.065$ and $\lambda \in [3, 8]$. Each model estimated separately for true $\lambda$ by MLE for simulated series, $T = 35,000$. 
Figure: Relative KL divergence of GAS-t relative to GARCH (solid) and GARCH-t (dashed): $1 - \text{KL(GAS-t)}/\text{KL(GARCH)}$
Relative KL divergence

Figure: RKL optimality regions for GAS-t and GARCH-t: $\lambda = 3$ and true $f_t \approx 1.2$. 

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Figure: CKL variation for GAS-t and GARCH-t: $\lambda = 3$. 
What about actual forecasting?

Forecasting the time-varying feature in the model is of key importance for the forecasting of the time series $y_t$.

This is the study of Koopman, Lucas and Scharth (2014) and the results are presented next.
<table>
<thead>
<tr>
<th>Distribution</th>
<th>Density</th>
<th>Link function</th>
</tr>
</thead>
<tbody>
<tr>
<td>Poisson</td>
<td>$\frac{\lambda_t^y}{y_t!} e^{-\lambda_t}$</td>
<td>$\lambda_t = \exp(\alpha_t)$</td>
</tr>
<tr>
<td>Neg. Binomial</td>
<td>$\frac{\Gamma(k_1+y_t)}{\Gamma(k_1)\Gamma(y_t+1)} \left( \frac{k_1}{k_1+\lambda_t} \right)^{k_1} \left( \frac{\lambda_t}{k_1+\lambda_t} \right)^{y_t}$</td>
<td>$\lambda_t = \exp(\alpha_t)$</td>
</tr>
<tr>
<td>Exponential</td>
<td>$\lambda_t e^{-\lambda_t y_t}$</td>
<td>$\lambda_t = \exp(\alpha_t)$</td>
</tr>
<tr>
<td>Gamma</td>
<td>$\frac{1}{\Gamma(k_1)\beta_t^{k_1}} y_t^{k_1-1} e^{-y_t/\beta_t}$</td>
<td>$\beta_t = \exp(\alpha_t)$</td>
</tr>
<tr>
<td>Weibull</td>
<td>$\frac{k_1}{\beta_t} \left( \frac{y_t}{\beta_t} \right)^{k_1-1} e^{-(y_t/\beta_t)^{k_1}}$</td>
<td>$\beta_t = \exp(\alpha_t)$</td>
</tr>
<tr>
<td>Gaussian vol</td>
<td>$\frac{1}{\sqrt{2\pi}\sigma_t} e^{-y_t^2/2\sigma_t^2}$</td>
<td>$\sigma_t^2 = \exp(\alpha_t)$</td>
</tr>
<tr>
<td>Student’s t vol</td>
<td>$\frac{\Gamma(\nu+1)}{\sqrt{(\nu-2)\pi\Gamma(\nu/2)}\sigma_t} \left( 1 + \frac{y_t^2}{(\nu-2)\sigma_t^2} \right)^{-\nu+1/2}$</td>
<td>$\sigma_t^2 = \exp(\alpha_t)$</td>
</tr>
<tr>
<td>Gaussian copula</td>
<td>$\frac{1}{2\pi\sqrt{1-\rho^2}} \exp \left[ -\frac{z_{1t}^2+z_{2t}^2-2\rho_t z_{1t} z_{2t}}{2(1-\rho_t^2)} \right]$</td>
<td>$\rho_t = \frac{1-\exp(-\alpha_t)}{1+\exp(-\alpha_t)}$</td>
</tr>
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<td>$\frac{\Gamma(\nu+2)}{\Gamma(\nu/2)} \frac{1}{\sqrt{1-\rho^2}} \left[ 1 + \frac{z_{1t}^2+z_{2t}^2-2\rho_t z_{1t} z_{2t}}{\nu(1-\rho_t^2)} \right]^{-\nu+2/2}$</td>
<td>$\rho_t = \frac{1-\exp(-\alpha_t)}{1+\exp(-\alpha_t)}$</td>
</tr>
</tbody>
</table>
### Monte Carlo study

<table>
<thead>
<tr>
<th>Distribution</th>
<th>GAS $\nabla_t(\theta_t)$</th>
<th>ACM $\mathcal{I}_t(\theta_t)$</th>
<th>ACM $s_t$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Poisson</td>
<td>$\frac{y_t}{\lambda_t} - 1$</td>
<td>$\frac{1}{\lambda_t}$</td>
<td>$yt$</td>
</tr>
<tr>
<td>Neg. Binomial</td>
<td>$\frac{y_t}{\lambda_t} - \frac{k_1+y_t}{k_1+\lambda_t}$</td>
<td>$\frac{k_1}{\lambda_t(k_1+\lambda_t)}$</td>
<td>$yt$</td>
</tr>
<tr>
<td>Exponential</td>
<td>$\frac{1}{\lambda_t} - y_t$</td>
<td>$\frac{1}{\lambda_t^2}$</td>
<td>$yt$</td>
</tr>
<tr>
<td>Gamma</td>
<td>$\frac{y}{\theta_t^2} - \frac{k_1}{\beta_t}$</td>
<td>$\frac{k}{\beta_t^2}$</td>
<td>$yt / k_1$</td>
</tr>
<tr>
<td>Weibull</td>
<td>$\frac{k_1}{\beta_t} \left[ \left( \frac{y_t}{\beta_t} \right)^{k_1} - 1 \right]$</td>
<td>$\left( \frac{k_1}{\beta_t} \right)^2$</td>
<td>$\frac{yt}{\Gamma(1+k_1^{-1})}$</td>
</tr>
<tr>
<td>Gaussian vol</td>
<td>$\frac{1}{2\sigma_t^2} \left( \frac{y_t^2}{\sigma_t^2} - 1 \right)$</td>
<td>$\frac{1}{2\sigma_t^4}$</td>
<td>$y_t^2$</td>
</tr>
<tr>
<td>Student’s t vol</td>
<td>$\frac{1}{2\sigma_t^2} \left( \omega_t y_t^2 \right) - 1)$</td>
<td>$\frac{\nu}{2(\nu+3)\sigma_t^4}$</td>
<td>$y_t^2$</td>
</tr>
<tr>
<td></td>
<td>$\omega_t = \frac{\nu+1}{(\nu-2)+y_t^2/\sigma_t^2}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Gaussian cop</td>
<td>$\frac{(1+\rho^2)(\hat{z}<em>{1,t}-\rho_t) - \rho_t(\hat{z}</em>{2,t}-2)}{(1-\rho^2)^2}$</td>
<td>$\frac{1+\rho_t^2}{(1-\rho_t^2)^2}$</td>
<td>$z_{1,t}z_{2,t}$</td>
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<td>$\frac{(\nu+2+\nu\rho_t^2)}{(\nu+4)(1-\rho_t^2)^2}$</td>
<td>$z_{1,t}z_{2,t}$</td>
</tr>
<tr>
<td></td>
<td>$\omega_t = \frac{\nu+2}{\nu+\frac{2}{\omega_t \hat{z}<em>{2,t}-2\rho_t \hat{z}</em>{1,t}}} - \rho_t^2$</td>
<td></td>
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<tr>
<td>Model Type</td>
<td>Distribution</td>
<td>$\delta, d$</td>
<td>$\phi, b$</td>
</tr>
<tr>
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</tr>
<tr>
<td>Count</td>
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<td>0.00</td>
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<td>0.00</td>
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<tr>
<td>Intensity</td>
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<td>0.00</td>
<td>0.98</td>
</tr>
<tr>
<td>Duration</td>
<td>Gamma</td>
<td>0.00</td>
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<tr>
<td>Volatility</td>
<td>Gaussian</td>
<td>0.00</td>
<td>0.98</td>
</tr>
<tr>
<td>Volatility</td>
<td>Student’s $t$</td>
<td>0.00</td>
<td>0.98</td>
</tr>
<tr>
<td>Copula</td>
<td>Gaussian</td>
<td>0.02</td>
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</table>
Monte Carlo study

We consider these nine observation densities. The autoregressive state equation completes the specifications of all parameter-driven models.

We draw 1,000 time series realisations, \( n = 4,000 \) for each DGP. In each simulation, we use the first 2,000 observations to estimate the parameters for the following model specifications.

1. the correctly specified state space model;
2. the GAS model based on the same conditional observation density as the DGP
3. the ACM model for the corresponding specification;
4. in the case of the exponential, gamma, Weibull, and Gaussian models, a robust variant of the GAS and ACM specification.
We compute one-step ahead predictions for the next 2,000 values of $\tilde{f}_t$ given the parameter values estimated from the first 2,000 observations $y_t$.

We therefore consider two million ($2,000 \times 1,000$) forecasts for each specification.

We measure the accuracy by means of the mean squared error (MSE), in levels and relative to the MSE of the state space model.

We compute the MSE across the two million forecasts of $\tilde{f}_t$. 
### Simulation results

#### DGP by state space model

<table>
<thead>
<tr>
<th>Distribution</th>
<th>State Space</th>
<th>GAS</th>
<th>ACM</th>
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## Simulation results

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Empirical Study

We have daily and high-frequency prices for twenty stocks from the Dow Jones index (January 1993 – June 2012) and five major stock indices between (January 1996 – October 2012).

Parameter estimation for all eight models is based on daily close-to-close returns.

We compute one-step ahead forecasts starting in 2001 and 2004 for the stocks and indices.

For each model, parameters are re-estimated every three months, expanding window, incl. all previous daily returns.

The precision of the forecasts from a model is evaluated by comparing the volatility forecasts with the daily realised volatilities as measured from high-frequency data.
Models in empirical study

1. SV
2. GAS
3. GARCH
4. EGARCH
5. SV with leverage
6. GAS with leverage
7. GJR : GARCH with leverage
8. EGARCH with leverage
Relative variance of the residuals of Mincer-Zarnowitz regressions of the realised volatilities

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