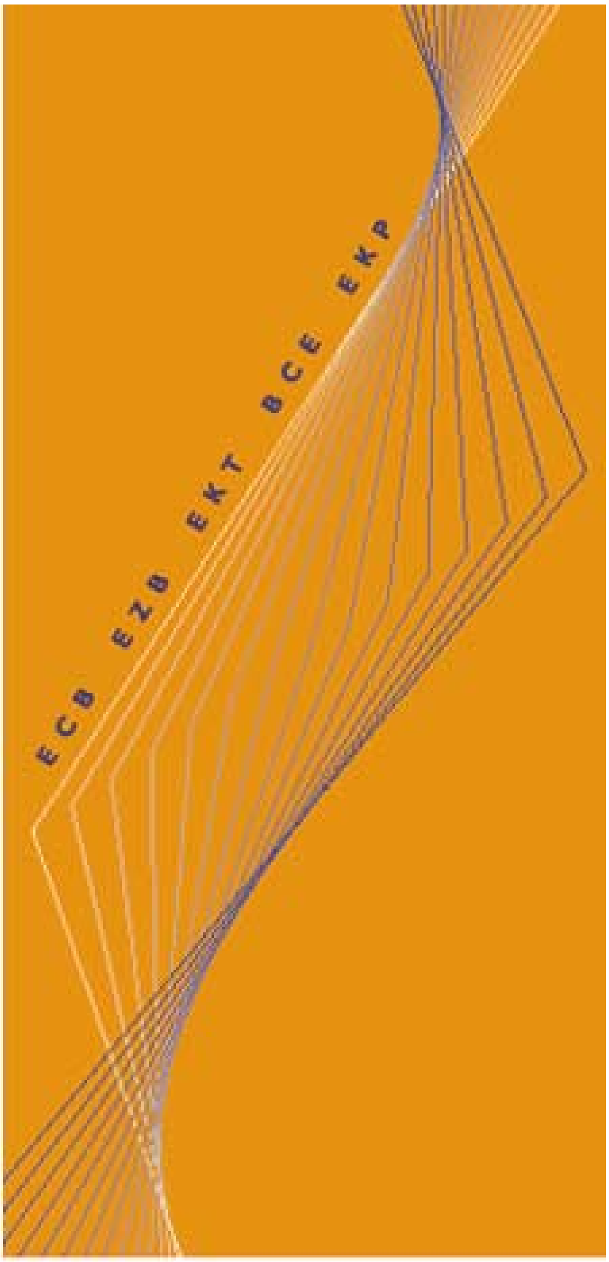
EUROPEAN CENTRAL BANK

WORKING PAPER SERIES



WORKING PAPER NO. 62

SPECTRAL BASED METHODS TO IDENTIFY COMMON TRENDS AND COMMON CYCLES

BY G. CAMBA MENDEZ AND G. KAPETANIOS

April 2001

EUROPEAN CENTRAL BANK

WORKING PAPER SERIES

WORKING PAPER NO. 62

SPECTRAL BASED METHODS TO IDENTIFY COMMON TRENDS AND COMMON CYCLES

BY G. CAMBA MENDEZ* AND G. KAPETANIOS**

April 2001

* *European Central Bank, Kaiserstrasse 29, D-60311, Frankfurt am Main, Germany, email: gonzalo.camba-mendez@ecb.int.*

** *Bank of England, Monetary Analysis Division 4 (HO-2), Threadneedle Street, EC2R 8AH, London, United Kingdom, email: George.Kapetanios@bankofengland.co.uk.* *The opinions expressed in this paper are those of the authors and do not necessarily reflect the views of the European Central Bank or the Bank of England.*

Address	**Kaiserstrasse 29**
	D-60311 Frankfurt am Main
	Germany
Postal address	**Postfach 16 03 19**
	D-60066 Frankfurt am Main
	Germany
Telephone	**+49 69 1344 0**
Internet	**http://www.ecb.int**
Fax	**+49 69 1344 6000**
Telex	**411 144 ecb d**

ISSN 1561-0810

Contents

Spectral Based Methods to Identify Common Trends and Common Cycles

Gonzalo Camba-Mendez* and George Kapetanios†

April 23, 2001

Abstract

The rank of the spectral density matrix conveys relevant information in a variety of modelling scenarios. Phillips (1986) showed that a necessary condition for cointegration is that the spectral density matrix of the innovation sequence at frequency zero is of a reduced rank. In a recent paper Forni and Reichlin (1998) suggested the use of a generalised dynamic factor model to explain the dynamics of a large set of macroeconomic series. Their method relied also on the computation of the rank of the spectral density matrix. This paper provides formal tests to estimate the rank of the spectral density matrix at any given frequency. The tests of rank at frequency zero are tests of the null of 'cointegration', complementary to those suggested by Phillips and Ouliaris (1988) which test the null of 'no cointegration'.

Keywords: Tests of Rank, Spectral Density Matrix, Canonical Correlations.
JEL Classification: C12, C15, C32

*European Central Bank, Kaiserstrasse 29, D-60311, Frankfurt am Main, Germany, email: gonzalo.camba-mendez@ecb.int .

†Bank of England, Monetary Analysis Division 4 (HO-2), Threadneedle Street, EC2R 8AH, London, United Kingdom, email: George.Kapetanios@bankofengland.co.uk. The opinions expressed in this paper are those of the authors and do not necessarily reflect the views of the European Central Bank or the Bank of England.

1 Introduction

The equivalence of time-domain and frequency-domain analysis of time series is well documented in the statistical and econometric literature. Nevertheless, the use of spectral densities is by far less widespread than the use of covariances in the econometric analysis of time series. This paper aims to apply both well known and new techniques of rank determination of matrices to the determination of the rank of the spectral density matrix. As the discussion below shows, the need for such techniques is clear from the existing econometric literature.

The idea that the movements in a number of economic variables can be represented by relatively few driving forces was first suggested by Stone (1947). Stone's use of principal components was purely static. Since then there has been a considerable amount of work using alternative data reduction techniques to promote parsimony in dynamic model specification. In a recent paper Forni and Reichlin (1998) suggested the use of a generalised dynamic factor model to describe the dynamics of sectoral industrial output and productivity for the US economy from 1958 to 1986. The number of common shocks driving those series is equal to the rank of their spectral density matrix. The foundations for this result are to be found in the literature on dynamic principal components, see Brillinger (1981). This issue is further explored in Forni, Hallin, Lippi, and Reichlin (1999a) and Forni, Hallin, Lippi, and Reichlin (1999b) where a 'generalised dynamic factor' model, novel to the literature, is proposed. Forni and Reichlin (1998) pointed that no standard test of the rank of the spectral density matrix was available, and consequently this issue was partially sidestepped.

Phillips (1986) showed that a necessary condition for cointegration is that the spectral density matrix of the innovation sequence of an I(1) multivariate process has a reduced rank at frequency zero. Phillips and Ouliaris (1988) suggested two procedures for detecting the presence of cointegration. The drawback of their method was that they were tests of the null of 'no cointegration'. Namely a test of the hypothesis that the r smallest eigenvalues are greater than zero. The tests of rank of the spectral density matrix suggested in this paper may be thought of, under certain conditions, as tests of the null of 'cointegration', i.e. tests of the null that the r smallest eigenvalues are equal to zero.

In order to apply standard tests of rank, it is helpful to think of the estimate of the spectral density matrix as an estimate of a covariance matrix between two newly defined complex variate random processes. Two alternative approaches for determining the rank of the spectral density matrix are presented. On the one hand, the Bartlett

(1947) procedure is applicable in this context because the problem can be recast in terms of canonical correlations. On the other hand, the Cragg and Donald (1996) approach is more general as it only requires that an estimate of that matrix exists having a normal asymptotic distribution with a covariance matrix whose rank is known. An information criterion method suggested by Akaike (1976), which builds upon the Bartlett (1947) procedure is also described.

Section 2 presents the analytical framework. Expressions for two alternative estimates of the spectral density matrix are provided here, together with some background material. The extensions of the Bartlett (1947) test of rank, Akaike (1976) information criterion method and Cragg and Donald (1996) test to the case of the spectral denstity matrix are described in sections 3 and 4. Section 5 concludes.

2 Background Theory

2.1 The Complex Multivariate Normal Distribution

In this paper references are made to the complex normal distribution. A q-random variable $\boldsymbol{y}_t$ with complex value components is complex multivariate normally distributed with mean $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Omega}$, and denoted as $N^C(\boldsymbol{\mu}, \boldsymbol{\Omega})$, if the $2q$-random variable with real components $(Re\ \boldsymbol{y}_t', Im\ \boldsymbol{y}_t')'$ is distributed as

$$N\left(\begin{bmatrix} Re\ \boldsymbol{\mu} \\ Im\ \boldsymbol{\mu} \end{bmatrix}, \frac{1}{2}\begin{bmatrix} Re\ \boldsymbol{\Omega} & -Im\ \boldsymbol{\Omega} \\ Im\ \boldsymbol{\Omega} & Re\ \boldsymbol{\Omega} \end{bmatrix}\right)$$

where Re and Im are operators which extract the real and imaginary part of a complex variate written in its cartesian form, and N denotes the multivariate normal distribution.

If a set of vector random variables, $\boldsymbol{y}_1, \ldots, \boldsymbol{y}_n$ are i.i.d zero mean complex multivariate normal with covariance $\boldsymbol{\Omega}$, then $\sum_{i=1}^{n} \boldsymbol{y}_i \bar{\boldsymbol{y}}_i{}'$ is said to have a complex Wishart distribution with n degrees of freedom, and is denoted by $W^C(n, \boldsymbol{\Omega})$.

2.2 Spectral Density Matrix Estimates

Denote a zero mean, wide sense stationary m-vector process by $\{\boldsymbol{x}_t\}_{t=1}^{\infty}$. The spectral density matrix of $\boldsymbol{x}_t$ is defined as the infinite Fourier transform of the covariance matrix function,

$$\boldsymbol{\Sigma}(\theta) = (2\pi)^{-1} \sum_{k=-\infty}^{\infty} \boldsymbol{\Gamma}_k e^{-ik\theta}$$

for $\theta \in [-\pi, \pi]$ where $\mathbf{\Gamma}_k = E\{\boldsymbol{x}_t \boldsymbol{x}'_{t-k}\}$. Given a sample of T observations an obvious estimate of the spectral density matrix is given by:

$$\overline{\mathbf{\Sigma}}(\theta) = (2\pi)^{-1} \sum_{k=-(T-1)}^{T-1} \hat{\mathbf{\Gamma}}_k e^{-ik\theta}$$

where $\hat{\mathbf{\Gamma}}_k = \frac{1}{T} \sum_{t=1}^{T-|k|} \boldsymbol{x}_t \boldsymbol{x}'_{t-k}$. $2\pi\overline{\mathbf{\Sigma}}(\theta)$ is known as the periodogram. It is well known that the periodogram is an inconsistent but asymptotically unbiased estimate of the spectral density matrix, multiplied by 2π and is asymptotically distributed as a complex Wishart variable with 1 degree of freedom. This suggests some form of smoothing can provide a consistent estimate of the spectral density matrix[1]. Two approaches are usually adopted. Firstly, methods that rely on 'smoothing' the autocovariance matrix function which take the form:

$$\hat{\mathbf{\Sigma}}(\theta) = \sum_{k=-M}^{M} \hat{\mathbf{\Gamma}}_k \omega(k, M) e^{-ik\theta} \tag{1}$$

where M is a bandwith parameter and $\omega(k, M)$ is a kernel or spectral window satisfying certain regularity conditions, (see, e.g. Fuller (1996)). As long as $M \to \infty$ and $T^{-1}M \to 0$ as $T \to \infty$, $\hat{\mathbf{\Sigma}}(\theta)$ provides a consistent estimate of $\mathbf{\Sigma}(\theta)$, see, e.g., Fuller (1996, pp. 382-383).

Secondly methods that rely on 'smoothing' the periodogram itself over the frequencies, i.e. averaging adjacent frequency ordinates. These estimates take the form,

$$\tilde{\mathbf{\Sigma}}(\theta) = \frac{1}{2M+1} \sum_{k=-M}^{M} \overline{\mathbf{\Sigma}}(\theta + k/T) \tag{2}$$

For finite M this estimate is still inconsistent, asymptotically unbiased for the spectral density matrix and asymptotically distributed as $(2M+1)^{-1} W^C(2M+1, \mathbf{\Sigma}(\theta_j))$, (see Brillinger (1981, pp. 245)). This is the simplest form of a smoothed periodogram estimate for the spectral density matrix. Different weights can be assigned to the periodogram coordinates $\bar{\mathbf{\Sigma}}\,(\theta + k/T)$, see Brillinger (1981, Chapter 7). If we allow $M \to \infty$ as $T \to \infty$ but impose $M/T \to 0$ we get a consistent estimate. In particular we obtain the result that $\sqrt{M}(vec(\tilde{\mathbf{\Sigma}}(\theta)) - vec(\mathbf{\Sigma}(\theta)))$ is asymptotically complex normal[2] with a covariance matrix whose element giving the covariance between $\mathbf{\Sigma}_{i,j}(\theta)$ and $\mathbf{\Sigma}_{u,v}(\theta)$is given by $\mathbf{\Sigma}_{i,j}(\theta)\mathbf{\Sigma}_{u,v}(\theta)$, where $\mathbf{\Sigma}_{i,j}(\theta)$ is the (i, j)-th element of $\mathbf{\Sigma}(\theta)$. We will denote this covariance matrix by $\boldsymbol{V}$ and its estimate, obtained by using the estimated spectral density matrix, by $\tilde{\boldsymbol{V}}$. More details may be found in e.g. Brillinger (1981, pp. 262) or Brockwell

[1]As we are mainly interested with the rank of the spectral density matrix, in the rest of the discussion we drop the normalising constant 2π.

[2]Note that this result crucially depends on regulating the dependence properties of the series. For more details see Brillinger (1981, Chapter 2).

and Davis (1991, pp. 447). Alternative strategies to select the weighting parameters to compute $\hat{\Sigma}(\theta)$ and $\tilde{\Sigma}(\theta_j)$ are well documented in the literature, see Priestley (1981) or Brockwell and Davis (1991) for further details.

3 Bartlett (1947) test of rank: Applicable to $\hat{\Sigma}(\theta)$

A well known result in canonical correlation analysis is that given two random vector series $\boldsymbol{x}_{1t}$ and $\boldsymbol{x}_{2t}$, the rank of their covariance matrix is equal to the number of nonzero canonical correlations. Further details are in Anderson (1984). Note that for a given bandwith M and a chosen kernel $\omega(k, M)$, the spectral density matrix estimate $\hat{\Sigma}(\theta)$ in (2) can be written as $\hat{\Sigma}(\theta) = \hat{\Pi}\Psi(\theta)$ where

$$\hat{\Pi} = \begin{bmatrix} \hat{\Gamma}_{-M} & \dots & \hat{\Gamma}_0 & \dots & \hat{\Gamma}_M \end{bmatrix} \qquad \Psi(\theta) = \begin{bmatrix} \boldsymbol{I}_m \times \phi^{\theta}_{-M} \\ \dots \\ \boldsymbol{I}_m \times \phi^{\theta}_0 \\ \dots \\ \boldsymbol{I}_m \times \phi^{\theta}_M \end{bmatrix}$$

$\boldsymbol{I}_m$ is an identity matrix of order m, and $\phi^{\theta}_s = \omega(s, M)e^{is\theta}$.

The interesting feature is that the spectral density matrix estimate is equivalent to the estimate of the covariance matrix between two complex variates, and defined as $\Sigma(\theta) = E\{\boldsymbol{x}_{1t}\boldsymbol{x}'_{2t}\}$, where $\boldsymbol{x}_{1t} = \boldsymbol{x}_t$ and $\boldsymbol{x}_{2t} = \Psi(\theta)'\left(\boldsymbol{x}'_{t+M}, \dots, \boldsymbol{x}'_t, \dots, \boldsymbol{x}'_{t-M}\right)'$. This implies that testing for the rank of $\hat{\Sigma}(\theta)$ above is equivalent to testing for the number of positive canonical correlations between $\boldsymbol{x}_{1t}$ and $\boldsymbol{x}_{2t}$. To compute the canonical correlations between $\boldsymbol{x}_{1t}$ and $\boldsymbol{x}_{2t}$ we proceed as follows. Define the matrices $\boldsymbol{X}_1$, $\boldsymbol{X}_2$ and $\Psi(\theta)$ as:

$$\boldsymbol{X}_1 = \begin{bmatrix} \boldsymbol{x}'_1 \\ \dots \\ \boldsymbol{x}'_M \\ \boldsymbol{x}'_{M+1} \\ \dots \\ \boldsymbol{x}'_{T-M} \\ \boldsymbol{x}'_{T-M+1} \\ \dots \\ \boldsymbol{x}'_T \end{bmatrix} \quad \boldsymbol{X}_2 = \begin{bmatrix} \boldsymbol{x}'_{M+1} & \dots & \boldsymbol{x}'_1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ \boldsymbol{x}'_{(2\times M)} & \dots & \boldsymbol{x}'_M & \dots & 0 \\ \boldsymbol{x}'_{(2\times M)+1} & \dots & \boldsymbol{x}'_{M+1} & \dots & \boldsymbol{x}'_1 \\ \dots & \dots & \dots & \dots & \dots \\ \boldsymbol{x}'_T & \dots & \boldsymbol{x}'_{T-M} & \dots & \boldsymbol{x}'_{T-(2\times M)} \\ 0 & \dots & \boldsymbol{x}'_{T-M+1} & \dots & \boldsymbol{x}'_{T-(2\times M)+1} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \boldsymbol{x}'_T & \dots & \boldsymbol{x}'_{T-M} \end{bmatrix}$$

Compute the QR decomposition of the matrices $\boldsymbol{X}_1$ and $\boldsymbol{X}_2\Psi(\theta)$ given above, i.e. $\boldsymbol{X}_1 = \boldsymbol{Q}_1\boldsymbol{R}_1$ and $\boldsymbol{X}_2\Psi(\theta) = \boldsymbol{Q}_2\boldsymbol{R}_2$. The canonical correlations between the vectors $\boldsymbol{x}_{1t}$ and $\boldsymbol{x}_{2t}$, are the singular values of $\boldsymbol{Q}'_1\boldsymbol{Q}_2$. We denote the canonical correlations as ρ_i, $i = 1, \dots, m$. Bartlett (1947) provided a criterion for testing the null hypothesis that the last $m - r^*$

canonical correlations are zero, i.e., $H_{r^*} : \rho_{r^*+1} = \cdots = \rho_m = 0$. Under the null hypothesis and assuming stationarity of the input-output multivariate system

$$BA = \left[\frac{2m+1}{2} - T\right] \ln\prod_{i=r^*+1}^{m}\left(1 - \hat{\rho}_i^2\right) \xrightarrow{d} \chi^2_{(m-r^*)^2}$$

Fujikoshi (1974) proved that this test procedure is based on the likelihood ratio method. His results can easily be extended to complex random variables providing justification for the application of the method in the current context. Bartlett's test was developed under independence and normality assumptions, but his result remains valid asymptotically following arguments by Kohn (1979) on the likelihood ratio tests for dependent observations.

3.1 Akaike (1976) Information Criterion

Akaike (1974) and Akaike (1976) showed that the number of linearly independent components of the projections of the previously defined $\boldsymbol{x}_{1t}$ onto the linear space spanned by the components of $\boldsymbol{x}_{2t}$ is identical to the number of nonzero canonical correlations between $\boldsymbol{x}_{1t}$ and $\boldsymbol{x}_{2t}$. When $\boldsymbol{x}_t$ is Gaussian, canonical correlation analysis between $\boldsymbol{x}_{1t}$ and $\boldsymbol{x}_{2t}$ is equivalent to maximum likelihood estimation of the linear model: $\boldsymbol{x}_{1t} = \boldsymbol{\Psi}\boldsymbol{x}_{2t} + \boldsymbol{\varepsilon}_t$, see Anderson (1984). The number of free parameters for this model is: $F(r^*) = \{[s_1(s_1+1)]/2\} + \{[s_2(s_2+1)]/2\} + r^*(s_1 + s_2 - r^*)$ where s_1 denotes the dimension of the vector $\boldsymbol{x}_{1t}$ and s_2 denotes the dimension of $\boldsymbol{x}_{2t}$. The first two terms are the number of free parameters of the covariance matrices of $\boldsymbol{x}_{1t}$ and $\boldsymbol{x}_{2t}$ respectively, and the last term gives the number of free parameters in matrix $\boldsymbol{\Psi}$. Akaike (1976) defined an information criterion for model fitting, and by extension rank determination, as:

$$AIC(r^*) = T \ln \prod_{i=1}^{r^*} (1 - \hat{\rho}_i^2) + 2F(r^*)$$

where $\hat{\rho}_i$ are the estimated canonical correlation coefficients previously defined. This criterion penalises models with a large number of parameters, and by extension large rank, and favours parsimonious representations. Schwarz (1978) suggested an alternative penalty on increasing the number of parameters. The information criterion suggested by Schwarz (1978) amounts to replace $2F(r^*)$ above by $\ln(T)F(r^*)$. The penalty used by this method is much more severe than that used by AIC.

4 Cragg and Donald test (1996)

We give an account of the general test of rank proposed by Cragg and Donald (1996) before discussing modifications for the problem at hand. For a general matrix $\boldsymbol{\Delta}$, the procedure proposed by Cragg and Donald (1996) is based on the transformation of the matrix $\boldsymbol{\Delta}$ using Gaussian elimination with complete pivoting[3]. r^* steps of Gaussian elimination with complete pivoting on matrix $\boldsymbol{\Delta}$ amounts to the following operations:

$$\boldsymbol{Q}_{r^*}\boldsymbol{R}_{r^*}\boldsymbol{Q}_{r^*-1}\boldsymbol{R}_{r^*-1}\ldots\boldsymbol{Q}_1\boldsymbol{R}_1\boldsymbol{\Delta}\boldsymbol{C}_1\ldots\boldsymbol{C}_{r^*-1}\boldsymbol{C}_{r^*}=\begin{bmatrix}\boldsymbol{\Delta}_{11}(r^*) & \boldsymbol{\Delta}_{12}(r^*)\\ \mathbf{0} & \boldsymbol{\Delta}_{22}(r^*)\end{bmatrix}$$

where $\boldsymbol{R}_i$ and $\boldsymbol{C}_i$ are pivoting matrices for step i and $\boldsymbol{Q}_i$ are Gauss transformation matrices. The pivoting matrices used to perform the first r^* steps of Gaussian elimininination are applied to $\boldsymbol{\Delta}$ to obtain the following relation

$$\boldsymbol{R}_{r^*}\boldsymbol{R}_{r^*-1}\ldots\boldsymbol{R}_1\boldsymbol{\Delta}\boldsymbol{C}_1\ldots\boldsymbol{C}_{r^*-1}\boldsymbol{C}_{r^*}=\boldsymbol{R}\boldsymbol{\Delta}\boldsymbol{C}=\boldsymbol{F}=\begin{bmatrix}\boldsymbol{F}_{11}(r^*) & \boldsymbol{F}_{12}(r^*)\\ \boldsymbol{F}_{21}(r^*) & \boldsymbol{F}_{22}(r^*)\end{bmatrix}$$

where $\boldsymbol{F}$ is partitioned accordingly, i.e. $\boldsymbol{F}_{11}(r^*)$ is of dimension $r^*\times r^*$. Note that in this case $\boldsymbol{F}_{11}(r^*)$ has full rank, under the null hypothesis that r^* is equal to the true rank. It then follows, see Cragg and Donald (1996), that $\boldsymbol{F}_{22}(r^*)-\boldsymbol{F}_{21}(r^*)\boldsymbol{F}_{11}^{-1}(r^*)\boldsymbol{F}_{12}(r^*)=0$. The estimated counterpart of the above relation, i.e. $\hat{\boldsymbol{F}}_{22}-\hat{\boldsymbol{F}}_{21}\hat{\boldsymbol{F}}_{11}^{-1}\hat{\boldsymbol{F}}_{12}=\hat{\boldsymbol{\Lambda}}_{22}(r^*)$, may be used as a test statistic of the hypothesis that the rank of $\boldsymbol{\Delta}$ is r^*. Under regularity conditions, including the requirement that the covariance matrix of the asymptotically normally distributed matrix $\sqrt{T}vec(\hat{\boldsymbol{\Delta}}-\boldsymbol{\Delta})$, denoted by $\boldsymbol{V}$, has full rank, the following result can be shown, under H_0.

$$\sqrt{T}vec(\hat{\boldsymbol{\Lambda}}_{22}(r^*))\xrightarrow{d}N(\mathbf{0},\boldsymbol{\Upsilon}\boldsymbol{V}\boldsymbol{\Upsilon}')$$

where $\boldsymbol{\Upsilon}=\boldsymbol{\Phi}_2\otimes\boldsymbol{\Phi}_1$ and $\boldsymbol{\Phi}_1=\left[-\hat{\boldsymbol{F}}_{21}\hat{\boldsymbol{F}}_{11}^{-1}\ \ \boldsymbol{I}_{m-r^*}\right]\boldsymbol{R}$, $\boldsymbol{\Phi}_2=\left[-\hat{\boldsymbol{F}}_{12}'\hat{\boldsymbol{F}}_{11}^{-1'}\ \ \boldsymbol{I}_{m-r^*}\right]\boldsymbol{C}'$ and $\xrightarrow{d}$ denotes convergence in distribution. Then,

$$\hat{\xi}=T vec\ \hat{\boldsymbol{\Lambda}}_{22}(r^*)'(\hat{\boldsymbol{\Upsilon}}\hat{\boldsymbol{V}}\hat{\boldsymbol{\Upsilon}}')^{-1}vec\ \hat{\boldsymbol{\Lambda}}_{22}(r^*)\xrightarrow{d}\chi^2_{(m-r^*)^2}$$

where $\hat{\boldsymbol{\Upsilon}}$ and $\hat{\boldsymbol{V}}$ are the sample estimates of $\boldsymbol{\Upsilon}$ and $\boldsymbol{V}$ and χ^2_l denotes the χ^2 distribution with l degrees of freedom. The procedure uses the inverse of the estimated asymptotic covariance matrix of $\boldsymbol{\Delta}$, but this may not be available if the covariance matrix is of reduced rank. However, Camba-Mendez and Kapetanios (2000) proved that the use of the Moore-Penrose inverse, denoted as $(\hat{\boldsymbol{\Upsilon}}\hat{\boldsymbol{V}}\hat{\boldsymbol{\Upsilon}}')^+$, is valid if the rank of $\boldsymbol{V}$ is known and $rk[\hat{\boldsymbol{V}}]=rk[\boldsymbol{V}]$, $\forall T>T_0$, for some finite T_0. Note that by construction the estimate of the asymptotic covariance matrix estimator of $\tilde{\boldsymbol{\Sigma}}(\theta)$, $\hat{\boldsymbol{V}}$, has this property. A sequential application of the Cragg and Donald test of rank can provide a consistent estimate of

[3] For details on Gaussian elimination with complete pivoting see Cragg and Donald (1996) or Golub and Loan (1983).

the rank of $\mathbf{\Delta}$ if the significance level used in the test converges to zero as the number of observations tends to infinity (See Hosoya (1989)).

We note that the Gaussian elimination results underlying the Cragg and Donald test remain valid for complex matrices such as $\mathbf{\Sigma}(\theta)$ and its estimator $\tilde{\mathbf{\Sigma}}(\theta)$. To see that we simply note that the r-th step of Gaussian elimination simply involves complete pivoting so as to bring the largest elements of the matrix in absolute value to the r-th place in the diagonal and subsequent zeroing out of the elements below that diagonal elements. Therefore, an $m \times m$ complex matrix with rank r subject to r steps of Gaussian elimination will be transformed into a matrix whose lower RHS $m-r \times m-r$ submatrix will be made up of zeros. The estimated counterpart of such a submatrix will have a multivariate complex normal with zero mean as long as the matrix is multivariate complex normal before the application of Gaussian elimination. This implies that the real and imaginary parts of the, typically complex in the case of spectral density estimates, elements of this submatrix, will be normally distributed with known covariance matrix, leading to a standard χ^2 test just like in the case of the Cragg and Donald test with real matrices.

Similar results may hold for the estimator $\hat{\mathbf{\Sigma}}(\theta)$ conditional on an asymptotic normality result for it similar to that provided by Brillinger (1981, pp. 262) for $\tilde{\mathbf{\Sigma}}(\theta)$. We do not know of a statement of such a result in the literature, although it seems safe to conjecture that such a result holds.

5 Conclusion

This paper has formulated a variety of rank detemination procedures for the rank of the spectral density matrix at any frequency. Different estimators of the spectral density matrix have been considered. Both testing procedures and information criteria have been suggested and justified as valid rank determination methods. The need for such techiques becomes apparent in the econometric literature in areas such as multivariate factor models and cointegration. Phillips and Ouliaris (1988) suggested tests of the null of 'no cointegration' which amounted to a test of the hypothesis that the r smallest eigenvalues of the spectral density matrix of the innovation sequence at frequency zero are greater than zero. Phillips and Ouliaris (1990) expanded on the issue of choice of the null hypothesis in cointegration testing by pointing out that adopting the null hypothesis of cointegration may be more sensible from a methodological point of view given that cointegration is the focus of interest. However, it was also pointed out that standard test statistics based on the spectral density matrix provided inconsistent tests under the null hypothesis of no cointegration. This paper has described tests of the rank of the spectral density matrix which may serve, at frequency zero, as tests of the null of 'cointegration'. It is clear that, as long as a consistent estimate of the spectral density matrix of the innovation process exists and has an asymptotic complex normal distribution, the application of the Cragg and Donald test will provide a consistent testing procedure for cointegration. The tests of the rank of the spectral density matrix described in this paper are also relevant to identify the structure of a vector times series under the approach of dynamic principal components. Further use of the techniques may be envisaged in terms of restricting the dimensionality of cyclical components at individual frequencies, possibly motivated from economic theory, thereby extending the common cycle analysis discussed in, e.g., Vahid and Engle (1993) and Engle and Issler (1995).

References

AKAIKE, H. (1974): "Stochastic Theory of Minimal Realisations," *IEEE Transactions on Automatic Control*, AC-19(6), 667–674.

——— (1976): "Canonical Correlation Analysis of Time Series and the Use of an Information Criterion," in *System Identification*, ed. by R. Mehra, and D. Lainiotis. Academic Press.

ANDERSON, T. W. (1984): *Introduction to Multivariate Statistical Analysis.* John Wiley and Sons, New York.

BARTLETT, M. (1947): "Multivariate Analysis," *Journal of the Royal Statistical Society, Series B*, 9, 176–197.

BRILLINGER, D. (1981): *Time Series: Data Analysis and Theory.* Holden-Day, San Francisco.

BROCKWELL, P. J., AND R. A. DAVIS (1991): *Time Series: Theory and Methods.* Springer Series in Statistics, Springer, New York.

CAMBA-MENDEZ, G., AND G. KAPETANIOS (2000): "Testing the Rank of the Hankel matrix: A Statistical Approach," Forthcoming in IEEE Transactions on Automatic Control.

CRAGG, J. G., AND S. G. DONALD (1996): "On the Asymptotic Properties of LDU-Based Tests of the Rank of a Matrix," *Journal of the American Statistical Association*, 91(435), 1301–1309.

ENGLE, R., AND J. ISSLER (1995): "Estimating Common Sectoral Cycles," *Journal of Monetary Economics*, 35, 83–113.

FORNI, M., M. HALLIN, M. LIPPI, AND L. REICHLIN (1999a): "The Generalised Factor Model: Identification and Estimation," *Review of Economics and Statistics. Forthcoming.*

——— (1999b): "Reference Cycles: the NBER methodology revisited," NBER Working Paper.

FORNI, M., AND L. REICHLIN (1998): "Let's Get Real: A factor Analytical Approach to Disaggregated Business Cycle Dynamics," *Review of Economic Studies*, 65, 453–473.

FUJIKOSHI, Y. (1974): "The Likelihood Ratio Tests for the Dimensionality of Regression Coefficients," *Journal of Multivariate Analysis*, 4, 327–340.

FULLER, W. A. (1996): *Introduction to Statistical Time Series, 2nd Ed.* Wiley.

GOLUB, G. H., AND C. F. V. LOAN (1983): *Matrix Computation.* North Oxford Academic.

HOSOYA, Y. (1989): "Hierarchical Statistical Models and a Generalised Likelihood Ratio Test," *Journal of the Royal Statistical Society B*, 51(3), 435–448.

KOHN, R. (1979): "Asymptotic Estimation and Hypothesis Testing Results for Vector Linear Time Series Models," *Econometrica*, 47(4), 1005–1030.

PHILLIPS, P. C. B. (1986): "Understanding Spurious Regressions," *Journal of Econometrics*, 33, 311–340.

PHILLIPS, P. C. B., AND S. OULIARIS (1988): "Testing for cointegration using principal components methods," *Journal of Economic Dynamics and Control*, 12, 205–230.

——— (1990): "Asymptotic Properties of Residual Based Tests for Cointegration," *Econometrica*, 58(1), 165–193.

PRIESTLEY, M. B. (1981): *Spectral Analysis and Time Series.* Academic Press, London.

SCHWARZ, G. (1978): "Estimating the Dimension of a Model," *Annals of Statistics*, 6, 461–464.

STONE, J. (1947): "The interdependence of blocks of transactions," *Journal of the Royal Statistical Society Supplement*, 9.

VAHID, F., AND R. F. ENGLE (1993): "Common Trends and Common Cycles," *Journal of Applied Econometrics*, 8, 341–360.

European Central Bank Working Paper Series

1 "A global hazard index for the world foreign exchange markets" by V. Brousseau and F. Scacciavillani, May 1999.

2 "What does the single monetary policy do? A SVAR benchmark for the European Central Bank" by C. Monticelli and O. Tristani, May 1999.

3 "Fiscal policy effectiveness and neutrality results in a non-Ricardian world" by C. Detken, May 1999.

4 "From the ERM to the euro: new evidence on economic and policy convergence among EU countries" by I. Angeloni and L. Dedola, May 1999.

5 "Core inflation: a review of some conceptual issues" by M. Wynne, May 1999.

6 "The demand for M3 in the euro area" by G. Coenen and J.-L. Vega, September 1999.

7 "A cross-country comparison of market structures in European banking" by O. de Bandt and E. P. Davis, September 1999.

8 "Inflation zone targeting" by A. Orphanides and V. Wieland, October 1999.

9 "Asymptotic confidence bands for the estimated autocovariance and autocorrelation functions of vector autoregressive models" by G. Coenen, January 2000.

10 "On the effectiveness of sterilized foreign exchange intervention" by R. Fatum, February 2000.

11 "Is the yield curve a useful information variable for the Eurosystem?" by J. M. Berk and P. van Bergeijk, February 2000.

12 "Indicator variables for optimal policy" by L. E. O. Svensson and M. Woodford, February 2000.

13 "Monetary policy with uncertain parameters" by U. Söderström, February 2000.

14 "Assessing nominal income rules for monetary policy with model and data uncertainty" by G. D. Rudebusch, February 2000.

15 "The quest for prosperity without inflation" by A. Orphanides, March 2000.

16 "Estimating the implied distribution of the future short term interest rate using the Longstaff-Schwartz model" by P. Hördahl, March 2000.

17 "Alternative measures of the NAIRU in the euro area: estimates and assessment" by S. Fabiani and R. Mestre, March 2000.

18 "House prices and the macroeconomy in Europe: Results from a structural VAR analysis" by M. Iacoviello, April 2000.

19 “The euro and international capital markets” by C. Detken and P. Hartmann, April 2000.

20 “Convergence of fiscal policies in the euro area” by O. De Bandt and F. P. Mongelli, May 2000.

21 “Firm size and monetary policy transmission: evidence from German business survey data” by M. Ehrmann, May 2000.

22 “Regulating access to international large value payment systems” by C. Holthausen and T. Rønde, June 2000.

23 “Escaping Nash inflation” by In-Koo Cho and T. J. Sargent, June 2000.

24 “What horizon for price stability” by F. Smets, July 2000.

25 “Caution and conservatism in the making of monetary policy” by P. Schellekens, July 2000.

26 “Which kind of transparency? On the need for clarity in monetary policy-making” by B. Winkler, August 2000.

27 “This is what the US leading indicators lead” by M. Camacho and G. Perez-Quiros, August 2000.

28 “Learning, uncertainty and central bank activism in an economy with strategic interactions” by M. Ellison and N. Valla, August 2000.

29 “The sources of unemployment fluctuations: an empirical application to the Italian case” by S. Fabiani, A. Locarno, G. Oneto and P. Sestito, September 2000.

30 “A small estimated euro area model with rational expectations and nominal rigidities” by G. Coenen and V. Wieland, September 2000.

31 “The disappearing tax base: Is foreign direct investment eroding corporate income taxes?” by R. Gropp and K. Kostial, September 2000.

32 “Can indeterminacy explain the short-run non-neutrality of money?” by F. De Fiore, September 2000.

33 “The information content of M3 for future inflation” by C. Trecroci and J. L. Vega, October 2000.

34 “Capital market development, corporate governance and the credibility of exchange rate pegs” by O. Castrén and T. Takalo, October 2000.

35 “Systemic risk: A survey” by O. De Bandt and P. Hartmann, November 2000.

36 “Measuring core inflation in the euro area” by C. Morana, November 2000.

37 “Business fixed investment: Evidence of a financial accelerator in Europe” by P. Vermeulen, November 2000.

38 “The optimal inflation tax when taxes are costly to collect” by F. De Fiore, November 2000.

39 “A money demand system for euro area M3” by C. Brand and N. Cassola, November 2000.

40 “Financial structure and the interest rate channel of ECB monetary policy” by B. Mojon, November 2000.

41 “Why adopt transparency? The publication of central bank forecasts” by P. M. Geraats, January 2001.

42 “An area-wide model (AWM) for the euro area” by G. Fagan, J. Henry and R. Mestre, January 2001.

43 “Sources of economic renewal: from the traditional firm to the knowledge firm” by D. R. Palenzuela, February 2001.

44 “The supply and demand for eurosystem deposits – The first 18 months” by U. Bindseil and F. Seitz, February 2001.

45 “Testing the Rank of the Hankel matrix: a statistical approach” by G. Camba-Mendez and G. Kapetanios, February 2001.

46 “A two-factor model of the German term structure of interest rates” by N. Cassola and J. B. Luís, February 2001.

47 “Deposit insurance and moral hazard: does the counterfactual matter?” by R. Gropp and J. Vesala, February 2001.

48 “Financial market integration in Europe: on the effects of EMU on stock markets” by M. Fratzscher, March 2001.

49 “Business cycle and monetary policy analysis in a structural sticky-price model of the euro area” by M. Casares, March 2001.

50 “Employment and productivity growth in service and manufacturing sectors in France, Germany and the US” by T. von Wachter, March 2001.

51 “The functional form of the demand for euro area M1” by L. Stracca, March 2001.

52 “Are the effects of monetary policy in the euro area greater in recessions than in booms?” by G. Peersman and F. Smets, March 2001.

53 “An evaluation of some measures of core inflation for the euro area” by J.-L. Vega and M. A. Wynne, April 2001.

54 “Assessment criteria for output gap estimates” by G. Camba-Méndez and D. R. Palenzuela, April 2001.

55 “Modelling the demand for loans to the private sector in the euro area” by A. Calza, G. Gartner and J. Sousa, April 2001.

56 "Stabilization policy in a two country model and the role of financial frictions" by E. Faia, April 2001.

57 "Model-based indicators of labour market rigidity" by S. Fabiani and D. Rodriguez-Palenzuela, April 2001.

58 "Business cycle asymmetries in stock returns: evidence from higher order moments and conditional densities" by G. Perez-Quiros and A. Timmermann, April 2001.

59 "Uncertain potential output: implications for monetary policy" by M. Ehrmann and F. Smets, April 2001.

60 "A multi-country trend indicator for euro area inflation: computation and properties" by E. Angelini, J. Henry and R. Mestre, April 2001.

61 "Diffusion index-based inflation forecasts for the euro area" by E. Angelini, J. Henry and R. Mestre, April 2001.

62 "Spectral based methods to identify common trends and common cycles" by G. C. Mendez and G. Kapetanios, April 2001.